Econ 704 Discussion Section-Handout 1

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1. Introduction:

- **Name**: Emilio Cuilty, **Born in**: Chihuahua, Mexico, **Occupation**: Fifth year student of the PhD. program in Economics
- **Research Interests**: Behavioral Economics, Industrial Organization and applied Structural Econometrics.
- **Warnings:** Strong accent and not the best handwriting

2. Problems

1. Set Theory: Verify the following Identities:
   (a) $A \backslash B = A \backslash (A \cap B) = A \cap B^c$ ($A \backslash B$ "Typically we say A minus B", means the part of A that does not intersects with B
   Take an element x in $A \backslash B$ (we write $x \in A \backslash B$), that element is in A but is not in B ($x \in A$ and $x \notin B$). Hence x is not in the intersection of A and B ($x \notin A \cap B$). In other words x is in A but not in intersection with A and B, or $x \in A \backslash (A \cap B)$. This shows that $A \backslash B \subset A \backslash (A \cap B)$. Follow the same logic backwards to show that $A \backslash (A \cap B) \subset A \backslash B$. For the second equality take $x \in A \backslash B$ this is true if and only if (we use $\iff$) $x \in A$ and $x \notin B \iff x \in A$ and $x \in B^c \iff x \in A \cap B^c$
   (b) $B = (B \cap A) \cup (B \cap A^c)$
   Take $x \in B$, then either $x \in A$ or $x \in A^c$. If $x \in A$ then $x \in A \cap B$, and hence $x \in (B \cap A) \cup (B \cap A^c)$, thus $B \subset (B \cap A) \cup (B \cap A^c)$. Now take $x \in (B \cap A) \cup (B \cap A^c)$ then $x \in (B \cap A)$ or $x \in (B \cap A^c)$, if $x \in (B \cap A^c)$ then $x \in B$.
   (c) $B \backslash A = B \cap A^c$
   Same logic as in a
   (d) $A \cup B = A \cup (B \cap A^c)$
   Now $A \cup B = A \cup ((B \cap A) \cup (B \cap A^c))$(from (b)), then $A \cup B = A \cup (B \cap A^c) \cup A \cup (B \cap A) = A \cup (B \cap A)$
2. Probability Measures: Answer the following questions
   (a) If $P(A) = \frac{1}{3}$ and $P(B^c) = \frac{1}{4}$ can A and B be disjoint?
   No, note that $P(B) = \frac{3}{4}$, if A and B are disjoint then $P(A \cap B) = 0$, and thus $P(A \cup B) = P(A) + P(B) = \frac{13}{12} > 1$ which violates one of the axioms
   (b) If $P(A \cup B) = \frac{1}{3}$ and $P(A^c) = \frac{2}{3}$, is it the case that $P(B) \leq P(A)$?
   Yes, note that $P(A) = \frac{1}{3}$, thus $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{3}$, which implies that $P(B) = P(A \cap B)$, so $P(A) \geq P(B)$
   (c) Is it true that $P(A \cap B) \geq P(A) + P(B) - 1$
   Yes, Note that $P(A \cap B) = P(A) + P(B) - P(A \cup B)$, and since $P(A \cup B) \leq 1$, it follows that $(A \cap B) \geq P(A) + P(B) - 1$

3. Countable Probability Measures (Challenging): Prove the following: If P is a probability measure over a countable sample space S, then
   (a) $P(A) = \sum_{i=1}^{\infty} P(A \cap C_i)$ for any partitions $C_1, C_2, ...$;
   since $C_1, C_2, ..$ is a partition, $C_i \cap C_j = \emptyset$, for any j, i. Also $\overset{\infty}{\underset{i=1}{\cup}} C_i = S$, since it is a countable sample set, hence $A = A \cap S = A \cap \left(\overset{\infty}{\underset{i=1}{\cup}} C_i \right) = \bigcup_{i=1}^{\infty} (A \cap C_i)$, and since $C_i s$ are disjoint so are any $(A \cap C_i)$, thus $P\left(\bigcup_{i=1}^{\infty}(A \cap C_i)\right) = \sum_{i=1}^{\infty} P(A \cap C_i)$
   (b) $P(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$ for any sets $A_1,\ A_2, ...$
   First construct disjoint sets A_i^* such that $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} A_i^*$,In a clever way, that is let $A_1^* = A_1$ and $A*_i = A_i \backslash \left(\bigcup_{j=1}^{i-1} A_j \right)$ for the rest of the $i's$
   then $P(\cup_{i=1}^{\infty} A_i) = P(\cup_{i=1}^{\infty} A_i^*)$,
   Note that $A_i^* \cap A_k^* = \left\{ A_i \backslash \left(\bigcup_{j=1}^{i-1} A_j \right) \right\} \cap \left\{ A_k \backslash \left(\bigcup_{j=1}^{k-1} A_j \right) \right\}$ or $A_i^* \cap A_k^* = \left\{ A_i \cap \left(\bigcup_{j=1}^{i-1} A_j \right)^c \right\} \cap \left\{ A_k \cap \left(\bigcup_{j=1}^{k-1} A_j \right)^c \right\}$ or $A_i^* \cap A_k^* = \left\{ A_i \cap \left(\bigcap_{j=1}^{i-1} A_j^c \right) \right\} \cap \left\{ A_k \cap \left(\bigcap_{j=1}^{k-1} A_j^c \right) \right\}$ (De Morgan's Law) now if $i > k$, the first intersection will be contained in the set A_k^c, which will have an empty intersection with A_k. if $K > i$ similar logic. Thus the sets A_i^* are disjoint. and hence $P(\cup_{i=1}^{\infty} A_i) = P(\cup_{i=1}^{\infty} A_i^*) = \sum_{i=1}^{\infty} P(A_i^*)$, finally note that for every i, $A_i^* \subset A_i$, thus $P(A_i^*) \leqq P(A_i)$, thus $P(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$
4. Conditional Probability: Answer the following questions
   (a) In my town, it's rainy one third of the days. Given that it is rainy, there will be heavy traffic with probability $\frac{1}{2}$, and given that it is not rainy, there will be heavy traffic with probability $\frac{1}{4}$. If it's rainy and there is heavy traffic, I arrive late for work with probability $\frac{1}{2}$. On the other hand, the probability of being late is reduced to $\frac{1}{8}$ if it is not rainy and there is no heavy traffic. In other situations (rainy and no traffic, not rainy and traffic) the probability of being late is 0.25. You pick a random day.
      i. What is the probability that it's not raining and there is heavy traffic and I am not late?
      $P(R^c \cap H \cap L^c) = P(R)P(T|R^C)P(L^c|R^c \cap T) = \frac{2}{3} * \frac{1}{4} * \frac{3}{4} = \frac{1}{8}$
      ii. What is the probability that I am late?
      $P(L) = P(R \cap T \cap L) + P(R \cap T^c \cap L) + P(R^c \cap T \cap L) + P(R^c \cap T^c \cap L) = \frac{1}{12} + \frac{1}{24} + \frac{1}{24} + \frac{1}{16} = \frac{11}{48}$
      iii. Given that I arrived late at work, what is the probability that it rained that day?
      $P(R|L) = \frac{P(R \cap L)}{P(R)}$, $P(R \cap L) = P(R \cap L \cap T) + P(R \cap L \cap T^c) = \frac{1}{12} + \frac{1}{24} = \frac{1}{8}$, and by iii. $P(R|L) = \frac{\frac{1}{8}}{\frac{11}{48}} = \frac{6}{11}$
   (b) In a TV Game show, a contestant selects one of three doors; behind one of the doors there is a prize, and behind the other two there are no prizes. After the contestant selects a door, the game-show host opens one of the remaining doors, and reveals that there is no prize behind it. The host then asks the contestant whether they want to SWITCH their choice to the other unopened door, or STICK to their original choice.
      i. Is it probabilistically advantageous for the contestant to SWITCH doors, or is the probability of winning the prize the same whether they STICK or SWITCH?
      Without loss of generality, let events A, B, C correspond to the prize being behind the selected, opened, and remaining door respectively, let H_B denote the event that the host opens B. We want to compare $P(A|H_B)$ stick, with $P(C|H_B)$ swtich. We know that $P(A) = P(B) = P(C) = \frac{1}{3}$, and $P(H_B|A) = \frac{1}{2}$, and $P(H_B|B) = 0$ and $P(H_B|C) = 1$. Then by Bayes Rule

$$P(A|H_B) = \frac{P(H_B|A)P(A)}{P(H_B|A)P(A) + P(H_B|B)P(B) + P(H_B|C)P(C)} = \frac{\frac{1}{2} * \frac{1}{3}}{\frac{1}{2} * \frac{1}{3} + 0 * \frac{1}{3} + 1 * \frac{1}{3}} = \frac{1}{3}$$

      if you do $P(C|H_B) = \frac{2}{3}$, hence you are better off if you switch