Econ 704 Discussion Section-Handout 2

Handout Created by Jian Zhang

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1. Quick Review

- Independence of events: $A_1, A_2, \cdots, A_n$ are independent if $\Pr(\bigcap_{i \in I} A_i) = \Pi_{i \in I} \Pr(A_i)$ for any subset $I \subseteq \{1, 2, 3, \cdots, n\}$.
- Remarks:
  - It means that you need to check this formula for all possible sets I.
  - Pairwise independence does not imply independence.
- Random variables are **functions** defined on sample space S.
- Independence of random variables: $X_1, X_2, \cdots, X_n$ are independent if $\Pr(X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n) = \Pr(X_1 = x_1)\Pr(X_2 = x_2)\cdots\Pr(X_n = x_n)$ for any possible outcome $(x_1, x_2, x_3, \cdots, x_n)$.
- Remark:
  - You just need to check one equation, this is slightly different from events independence.
  - $\{X_1 = x_1, X_2 = x_2\}$ means $\{X_1 = x_1\}\bigcap\{X_2 = x_2\}$.
- Formulas:

$$E(aX + b) = aEX + b; \qquad E(X + Y) = E(X) + E(Y);$$

$$Var(aX + b) = a^2 Var(X); \qquad Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y);$$

$$Cov(aX + b, cY + d) = acCov(X, Y);$$

2. Exercises

1. **(Independence of Events)** There are 4 balls labeled by 111, 100, 010, 001 in a urn. Suppose that you randomly pick a ball from this urn. Let A_k be the event "k-th digit of its label is 1" ($k = 1, 2, 3$).
   (a) Show that A_i and A_j are independent for $i \neq j$ (so-called "pairwise independent").
   $\Pr(A_1) = \Pr(A_2) = \Pr(A_3) = \frac{1}{2}$ and $\Pr(A_i \cap A_j) = \Pr(111 \text{ is chosen}) = 1/4$ for $i \neq j$. Therefore, $\Pr(A_i \cap A_j) = \Pr(A_i)\Pr(A_j)$
   (a) Are A_1, A_2 and A_3 independent?
   No. $\Pr(A_1 \cap A_2 \cap A_3) = \Pr(111 \text{ is chosen}) = 1/4 \neq \Pr(A_1)\Pr(A_2)\Pr(A_3) = 1/8$.
2. **(Independence of Events)** Suppose $\Pr(A) > 0$ and $\Pr(B) > 0$. If A and B are disjoint, could they be independent?
   No. A and B are disjoint, then $A \cap B = \emptyset$. Then $\Pr(A \cap B) = 0 \neq \Pr(A)\Pr(B)$

3. **(Independence of Variables)** Suppose X_1, X_2 and X_3 are discrete random variables. Prove that if X_1, X_2 and X_3 are independent, then X_1 and X_2 are independent.

X_1, X_2 and X_3 are independent $\Longrightarrow \Pr(X_1 = x_1, X_2 = x_2, X_3 = x_3) = \Pr(X_1 = x_1)\Pr(X_2 = x_2)\Pr(X_3 = x_3)$ for all realizations x_1, x_2, x_3. $\Longrightarrow \Pr(X_1 = x_1, X_2 = x_2) = \sum_{x_3} \Pr(X_1 = x_1, X_2 = x_2, X_3 = x_3) = \Pr(X_1 = x_1)\Pr(X_2 = x_2)\sum_{x_3} \Pr(X_3 = x_3)$, that is,

$$\Pr(X_1 = x_1, X_2 = x_2) = \Pr(X_1 = x_1)\Pr(X_2 = x_2).$$

Therefore, X_1 and X_2 are independent

4. **(Conditional Distribution of Random Variables)** Suppose X and Y are two random variables, their realizations can only be 0 or 1. Assume that $\Pr(X = 0) > 0$ and $\Pr(X = 1) > 0$.

   (a) Is it true that $\Pr(Y = 1) \geq \Pr(Y = 1|X = 0)$ or $\Pr(Y = 1) \geq \Pr(Y = 1|X = 1)$ holds?

   (b) Is it true that both $\Pr(Y = 1) > \Pr(Y = 1|X = 0)$ and $\Pr(Y = 1) > \Pr(Y = 1|X = 1)$ holds?

Note that

$$\Pr(Y = 1) = \Pr(Y = 1|X = 0)\Pr(X = 0) + \Pr(Y = 1|X = 1)\Pr(X = 1).$$

Therefore, $\Pr(Y = 1)$ is the weighted average of $\Pr(Y = 1|X = 0)$ and $\Pr(Y = 1|X = 1)$. Obviously, (a) is true but (b) is false.

5. **(The Law of Iterated Expectations)** Suppose that X_1, X_2 and X_3 are random variables with the same distribution. $E(X_1) = 3$. N is a random variable with $\Pr(N = 1) = \Pr(N = 2) = Pr(N = 3) = 1/3$ and independent of X_i for $i = 1, 2, 3$. Compute $E\big[\sum_{i=1}^{N} X_i\big]$

By the law of iterated expectations,

$$E[\sum_{i=1}^{N} X_i] = \sum_{j=1}^{3} E[\sum_{i=1}^{N} X_i|N = j]\Pr(N = j) = \frac{1}{3}\Big(E[X_1|N = 1] + E[\sum_{i=1}^{2} X_i|N = 2] + E[\sum_{i=1}^{3} X_i|N = 3]\Big).$$

Since N is independent of X_i, then $E[X_i|N = j] = E[X_i]$. Hence, the RHS of above formula is equal to

$$\frac{1}{3}\Big(E[X_1] + 2E[X_1] + 3E[X_3]\Big) = 2E[X_1] = 6.$$

6. **(Calculus is required!)** Suppose X and Y are random variables with $Var(X) > 0$. Find the optimal value of $a \in \mathbb{R}$ that minimizes $Var(Y - aX)$.

Let $f(a) = Var(Y - aX) = Var(Y) + a^2Var(X) - 2aCov(X, Y)$.

To minimize $f(a)$, we need $f'(a) = 0$, that is, $2aVar(X) - 2Cov(X, Y) = 0$.

Therefore, the optimal value of a is $Var(X)^{-1}Cov(X, Y)$.

7. **(Cauchy-Schwartz Inequality, Challenging)** Suppose X and Y are random variables with $Var(X) > 0$ and $Var(Y) > 0$. Prove that

   (a) $\big(Cov(X, Y)\big)^2 \leq Var(X) \cdot Var(Y)$.

   Let a be an arbitrary real number. Then

   $$0 \leq Var(Y - aX) = Var(Y) + a^2Var(X) - 2aCov(X, Y).$$

   Then we take $a = Var(X)^{-1}Cov(X, Y)$. Then above inequality becomes that

   $$0 \leq Var(Y) - Var(X)^{-1}\big(Cov(X, Y)\big)^2.$$

   Therefore, $\big(Cov(X, Y)\big)^2 \leq Var(X) \cdot Var(Y)$.

(b) X and Y's correlation ρ_{XY} satisfies that $-1 \leq \rho_{XY} \leq 1$

Note that

$$\rho_{XY}^2 = \frac{\big(Cov(X,Y)\big)^2}{Var(X) \cdot Var(Y)} \leq 1.$$

Therefore, $-1 \leq \rho_{XY} \leq 1$.