Econ 704 Discussion Section-Handout 3

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1. Review:

- Continuous random variables behave almost as discrete random variables.
- Most of the previous results hold by replacing the sum(s) with an integral(s) . HOWEVER transformations of continuous RVs are trickier and require additional care.
  - Univariate Case: if Y is a continuous R.V. with density f. Let $\mathcal{Y} = \{y : f(y) > 0\}$. X is transformation that comes from a function $\phi(\mathcal{Y})$, where $\phi(\mathcal{Y})$ is monotone and the derivative of its inverse is continuous . Then the density of $\mathcal{X}$ is given by
  $$g(x) = |(\phi^{-1})'(x)|f((\phi^{-1})(x))$$
  - Bivariate Case: Let X and Y be two random variables with joint density f, and let $\mathcal{A} = \{(x,y) : f(x,y) > 0\}$,;let $(U,V) = \phi(X,Y)$, where $\phi : \mathcal{A} \to \mathbb{R}^2$ is one to one with range $\mathcal{B} = \{\phi(x,y) : (x,y) \in \mathcal{A}\}$, and with differentiable inverse let $D\phi^{-1}(u,v) = \begin{pmatrix} \frac{\partial \phi_1^{-1}}{\partial u}(u,v) & \frac{\partial \phi_1^{-1}}{\partial v}(u,v) \\ \frac{\partial \phi_2^{-1}}{\partial u}(u,v) & \frac{\partial \phi_2^{-1}}{\partial v}(u,v) \end{pmatrix}$ be the Jacobian of $\phi^{-1}(u,v)$. Then the joint density function is defined on $\mathcal{B}$ by
  $$g(u,v) = |det\, D\phi^{-1}(u,v)|f(\phi^{-1}(u,v))$$

2. Problems

1. **(Bernoulli Trials)** Answer the following problems:
   (a) The Flow of traffic at certain street corners can be sometimes modeled as a sequence of Bernoulli trials by assuming that the probability of a car passing during any given second is a constant p and that there is no interaction between the passing cars at different seconds. If we treat seconds as indivisible time units, the Bernoulli model applies. Suppose that a pedestrian can cross the street only if no cars is to pas during the next 3 seconds. Find the probability that the pedestrian has to wait exactly 4 seconds before starting to cross

   In the seven seconds for the event, no car must pass in the last three seconds, an event with probability $(1-p)^3$. It also has to be true that in the fourth second there was a car, other wise the individual will crossed earlier, that happens with p probability. Finally in the first tree seconds there must be at least one car. So we have the following combinations

1	1	1	1	0	0	0	$(1-p)^3p^4$
1	1	0	1	0	0	0	$(1-p)^4p^3$
1	0	0	1	0	0	0	$(1-p)^5P^2$
0	1	1	1	0	0	0	$(1-p)^4p^2$
0	1	0	1	0	0	0	$(1-p)^5p^2$
0	0	1	1	0	0	0	$(1-p)^5p^2$

therefore

$$Pr(X = 4) = (1-p)^3 p^2 \left(p^2 + p(1-p) + (1-p) + 3(1-p)^2\right)$$

(b) A standard drug us known to be effective in 80% of the cases in which it is used. A new drug is tested on 100 patients and found to be effective in 85 cases. Is the new drug superior?

Let X =number of effective cases. If the new and old drugs are equally effective, then the probability that the new drug is effective on a case is .8.If the cases are independent then $X \sim binomial(100, .8)$, and $P(X \geq 85) = \sum_{x=85}^{100} \begin{pmatrix} 100 \\ 85 \end{pmatrix} .8^x .2^{100-x} = .1285$
So, even if the new drug is no better than the old, the chance of 85 or more effective cases is not too small. Hence, we cannot conclude the new drug is better.

2. **(Continuous Random Variables)**: Let X have the standard normal pdf find $E[X^2]$

using the definition of expected value we have that
$\int_{-\infty}^{\infty} x^2 \frac{1}{2\pi} e^{-\frac{x^2}{2}}\, dx = \frac{1}{2\pi}\left[-xe^{\frac{-x^2}{2}}|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}}\, dx\right] = \frac{1}{2\pi}(2\pi) = 1$

3. **(Univariate Transformation)** Answer the following problems:

(a) Let X have a pdf $f_X(x) = \frac{2}{9}(x+1)$ $-1 \leq x \leq 2$.

i. Find the pdf of $Y = X^2$

Note that X^2 is not a one to one transformation.
If $-1 \leq X \leq 1$, we could have $x = \pm\sqrt{y}$, however if $X > 1$, then the transformation is one to one

$$P(Y \leq y) = P(X^2 \leq y) = \begin{cases} P(-\sqrt{y} \leq X \leq \sqrt{y}) & if\, |x| \leq 1 \\ P(1 \leq X \leq \sqrt{y}) & if\, |x| > 1 \end{cases}$$

Then

$$P(Y \leq y) = \begin{cases} \int_{-\sqrt{y}}^{\sqrt{y}} \frac{2}{9}(x+1)\, dx & if\, |x| \leq 1 \\ \int_{1}^{\sqrt{y}} \frac{2}{9}(x+1)\, dx & if\, |x| > 1 \end{cases} = \begin{cases} \frac{4}{9}\sqrt{y} & if\, |x| \leq 1 \\ \frac{y}{9} + \frac{2\sqrt{y}}{9} + \frac{1}{9} & if\, |x| > 1 \end{cases}$$

Taking derivatives we get that

$$f_y(y) = \begin{cases} \frac{2}{9}\frac{1}{\sqrt{y}} & if\, y < 1 \\ \frac{1}{9} + \frac{1}{9}\frac{1}{\sqrt{y}} & if\, y > 1 \end{cases}$$

(b) Show that it is possible to use the transformation formula if the sets $A_1, A_2, .., A_k$ contain X and apply this extension to solve part (i) using $A_0 = \emptyset$ $A_1 = (-1, 1)$ $A_2 = $ (1,2)

. If the sets $B_1, B_2, .., B_k$ are a partition for the range of Y we can write

$$f_Y(y) = \sum_k f_Y(y) I(y \in B_k)$$

and do the transformation on each set B_k as the formula says, for A_1 and A_2 this is exactly as above.

(c) If the random variable X has a pdf

$$f(x) = \begin{cases} \frac{x-1}{2} & 1 < x < 3 \\ 0 & else \end{cases}$$

find a monotone function $\phi(x)$, such that $Y = \phi(x)$ has a uniform $(0,1)$ distribution

if $Y = \phi(x) = F(x)$, then $Y \sim U(0,1)$ then $\phi(x) = F(x) = \begin{cases} 0 & x \leq 1 \\ \frac{(x-1)^2}{4} & 1 < x < 3 \\ 1 & 3 \leq x \end{cases}$

4. **(Transformations and Expectations)** Prove that the "two-way" rule for expectations, $E[g(X)] = EY$, where $Y = g(X)$ and assume that $g(x)$ is an increasing function

   Since $g(x)$ is monotone, $E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx = \int_{-\infty}^{\infty} y f_X(g^{-1}(y)) g^{-1}(y)'\, dy = \int_{-\infty}^{\infty} y f_Y(y)\, dy = E[Y]$

5. **(Bivariate Distributions):** A pdf is defined by

$$f(x,y) = \begin{cases} C(x+2y) & if\, 0 < y < 1\, and\, 0 < x < 2 \\ 0 & else \end{cases}$$

   (a) Find the Value of C

   we know that the density should integrate to one, so $1 = \int_0^1 \int_0^2 C(x+2y)\, dx\, dy = \int_0^1 C(2+4y)\, dy = 4C$, so $C = \frac{1}{4}$

   (b) Find the marginal distribution of X

   Integrating over Y, we have that $\int_0^1 \frac{1}{4}(x+2y)\, dy = \frac{1}{4}(x+1)$. So
   $f_x(x) = \begin{cases} \frac{1}{4}(x+1) & 0 < x < 2 \\ 0 & else \end{cases}$

   (c) Find the joint cdf of X and Y

   Note that $F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\, dv\, du$, so the integral depends on the values of x and y. For example if $0 < y < 1\, and\, 0 < x < 2$
   then $F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\, dv\, du = \int_0^x \int_0^y \frac{1}{4}(u+2v)\, dv\, du = \frac{x^2 y}{8} + \frac{y^2 x}{4}$
   But if $1 \leq y$, $F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\, dv\, du = \int_0^x \int_0^1 \frac{1}{4}(u+2v)\, dv\, du = \frac{x^2}{8} + \frac{x}{4}$ and
   if $2 \leq x$, $F_{XY}(x,y) = \int_{-\infty}^{2} \int_{-\infty}^{y} f(u,v)\, dv\, du = \int_0^2 \int_0^x \frac{1}{4}(u+2v)\, dv\, du = \frac{y}{2} + \frac{y^2}{2}$
   so

$$F_{XY}(x,y) = \begin{cases} 0 & y < 0\, or\, x < 0 \\ \frac{x^2 y}{8} + \frac{y^2 x}{4} & 0 < y < 1\, and\, 0 < x < 2 \\ \frac{x^2}{8} + \frac{x}{4} & 1 \leq y\, and\, 0 < x < 2 \\ \frac{y}{2} + \frac{y^2}{2} & 0 < y < 1\, and\, 2 \leq x \\ 1 & else \end{cases}$$

6. **(Bivariate Distributions, Challenging):** Prove that for any random variables X and Y, $|Corr(X,Y)| = 1$ if an only if there exist numbers $a \neq 0$ and b such that $P(Y = aX + b) = 1$. If $Corr(X,Y) = 1$, then $a > 0$, and if $Corr(X,Y) = -1$ then $a < 0$

   Let $h(t) = E\left[(X - \mu_x)t + (Y - \mu_y)\right]^2$
   expanding we have that $h(t) = t^2 E(X - \mu_x)^2 + 2tE(X - \mu_x)E(Y - \mu_y) + E(Y - \mu_Y)^2$

that is $h(t) = t^2\sigma_X^2 + 2t\sigma_{XY} + \sigma_Y^2$,since it is the expected value of a non negative random variable, it is positive for all t

hence it has at most one positive root, by the general formula it should have a non positive discriminant.$4\sigma_{XY}^2 - 4\sigma_Y^2\sigma_X^2 \leq 0$

thus$-1 \leq \frac{\sigma_{XY}}{\sigma_Y\sigma_X} \leq 1$, and so $\frac{\sigma_{XY}}{\sigma_Y\sigma_X} = 1$ only if the discriminat is equal to zero. That is only if $h(t)$ has a single root,

but since $[(X-\mu_x)t + (Y-\mu_y)]^2 \geq 0$, then $h(t) = 0$ if and only if $P\left\{[(X-\mu_x)t + (Y-\mu_y)]^2 = 0\right\} = 1$,
or $P\{[(X-\mu_x)t + (Y-\mu_y)] = 0\} = 1$, that is $P(Y = -tX + \mu_X t + \mu_Y) = 1$, where t is the root of $h(t)$, and using the general quadratic formula we get that $t = -\sigma_{XY}/\sigma_X^2$. Thus, $a = -t$, has the same sign as $corr(X,Y)$, proving the proposition.

7. **(Bivariate Transformations)**: Answer the following questions

   (a) Use pdf from problem 5 to find the pdf of the random variable $Z = 9/(X+1)^2$

   note that Z is monotone on $0 < x < 2$, so $(X+1)^2 = \frac{9}{Z}$, and thus $X = \frac{3}{Z^{1/2}} - 1$, $\phi^{-1}(z)' = -\frac{3}{2z^{3/2}}$
   $f_z(Z) = \frac{3}{2z^{3/2}}\left(\frac{1}{4}(\frac{3}{Z^{1/2}})\right) = \frac{9}{8z^2}$, and the it goes from 1 to 9

   (b) X_1 and X_2 are Independent $Normal$s with $n(0,\sigma^2)$, find the joint distribution of Y_1 and Y_2. With $Y_1 = X_1^2 + X_2^2$ and $Y_2 = \frac{X_1}{\sqrt{Y_1}}$

   This transformation is not one to one, since you cannot tell the sign of X_2 from Y_1 and Y_2. So Partition the support of (X_1, X_2) into $A_0 = \{-\infty < x_1 < \infty, x = 0\}$, $A_1 = \{-\infty < x_1 < \infty, x > 0\}$ and $A_2 = \{-\infty < x_1 < \infty, x < 0\}$,then the support of $B = \{0 < y_1 < \infty,\ -1 < y_2 < 1\}$, so the inverse from $B \to A_1$is $x_1 = y_2\sqrt{y_1}$ and $x_2 = \sqrt{y_1 - y_1 y_2^2}$, with the Jacobian, $J_1 = \begin{pmatrix} \frac{1}{2}\frac{y_2}{\sqrt{y_1}} & \sqrt{y_1} \\ \frac{1}{2}\frac{\sqrt{1-y_2^2}}{\sqrt{y_1}} & \frac{y_2\sqrt{y_1}}{\sqrt{1-y_2^2}} \end{pmatrix}$, so$|J_1| = \frac{1}{2\sqrt{1-y_2^2}}$
   now the inverse from $B \to A_2$ is $x_1 = y_2\sqrt{y_1} and\ x_2 = -\sqrt{y_1 - y_1 y_2^2}$, so $J_2 = -J_1$, hence it has the same determinant.
   In theore we should sum both terms to get the pdf, but since they are the same,

   $$f_{Y_1Y_2}(y_1, y_2) = 2\left(\frac{1}{\pi\sigma^2}e^{-y_1/(2\sigma^2)}\frac{1}{2\sqrt{1-y_2^2}}\right)$$

   (c) Show that Y_1 and Y_2 from (b) are independent.

   Note that is possible to write each the joint pdf as the product of $\frac{1}{\pi\sigma^2}e^{-y_1/(2\sigma^2)}$ and $\frac{1}{2\sqrt{1-y_2^2}}$, and therefore they are independent.