Econ 704 Discussion Section-Handout 5

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1. Review:

- Distributional properties vs Sample path properties
  - Consider a sample space of $\Omega = \{Heads, Tails\}$, $P(H) = P(T) = \frac{1}{2}$
  - Now take a look at the following sequence of random variables $\{X_t\}_{t=1}^{\infty}$ $\{Y_t\}_{t=1}^{\infty}$, where

$$X_t(\omega) = \begin{cases} 1 & if\, \omega = H \\ -1 & if\, \omega = T \end{cases} \quad Y_t(\omega) = \begin{cases} 1 & if\, \omega = H\, \&\, t = odd\, or\, \omega = T\, \& t = even \\ -1 & if\, \omega = T\, \&\, t = odd\, or\, \omega = H\, \& t = even \end{cases}$$

  - The marginal distribution of $X_{21}(\omega)$ and $Y_{21}(\omega)$ are the same!
  - The joint distribution of $X_{21}(\omega)$, $X_{22}(\omega)$ is different from $Y_{21}(\omega)$, $Y_{22}(\omega)$
  - The sample paths are different!
- Consider a sequence of i.i.d random variables $\{X_i\}_{i=1}^{n}$ over a probability space $(\Omega, \mathcal{F}, P)$, suppose that $E[X_i] = \mu$
  - **The weak law of large numbers**: For all $\epsilon > 0$, $\underset{n\to\infty}{lim} P(\omega \in \Omega : \bar{X}_n(\omega) \in [\mu - \epsilon, \mu + \epsilon]) = 1$
  - **The strong law of large numbers:**

$$P\left(\omega \in \Omega : \underset{n\to\infty}{lim} \bar{X}_n(\omega) = \mu\right) = 1$$

- The Borel-Cantelli lemma: Consider a probability space $(\Omega, \mathcal{F}, P)$, let $\{A_i\}_{i=1}^{\infty}$ be a sequence of events
  - a) $\sum_{i=1}^{\infty} P(A_i) < \infty$ then $Pr\left(\bigcap\limits_{i=1}^{\infty}\bigcup\limits_{j=i}^{\infty} A_j\right) = 0$
  - b) if $\{A_i\}_{i=1}^{\infty}$ are independent, and $\sum_{i=1}^{\infty} P(A_i) = \infty$, then $Pr\left(\bigcap\limits_{i=1}^{\infty}\bigcup\limits_{j=i}^{\infty} A_j\right) = 1$

2. Problems

1. **(Borel Cantelli)** Answer the following problems:
   (a) Prove the second lemma
   It is sufficient to show that the event that theA_i's did not occur for infinitely many values of i has probability zero.
   That is $1 - Pr\left(\bigcap\limits_{i=1}^{\infty}\bigcup\limits_{j=i}^{\infty} A_j\right) = 0$, or $Pr\left(\left(\bigcap\limits_{i=1}^{\infty}\bigcup\limits_{j=i}^{\infty} A_j\right)^c\right) = 0$, $Pr\left(\bigcup\limits_{i=1}^{\infty}\bigcap\limits_{j=i}^{\infty} A_j^c\right) = 0$,
   which is equivalent at just to show $Pr\left(\bigcap\limits_{j=i}^{\infty} A_j^c\right) = 0$, and since $A_j's$ are independent, $Pr\left(\bigcap\limits_{j=i}^{\infty} A_j^c\right) = \prod\limits_{j=i}^{\infty} Pr(A_j^c)$

but $\prod_{j=i}^{\infty}(1-Pr(A_j)) \leq \prod_{j=i}^{\infty} exp(-Pr(A_j)) = exp\left(-\sum_{j=i}^{\infty} Pr(A_j)\right) = 0$

(b) Show that: If you have an infinite number of monkeys each hitting keys at random on typewriter keyboards then, with probability 1, one of them will type the complete works of William Shakespeare.

Let $A_n$the event that the n^{th} monkey types the complete works of William Shakespeare. Then if there are m characters on the keyboard and N characters in the complete works of Shakespeare, $P(A_n) = m^{-N}$ for each n. Furthermore the $A_n$are mutually independent. Note also that $\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} m^{-N} = \infty$ Hence by the second Borel-Cantelli lemma, infinitely many events of A_n occur i.e. infinitely many monkeys will type the complete works of William Shakespeare

(c) (Challenging): Let $\Omega = [0,1]$, $X_n(\omega)$ be a sequence of Random Variables, with . $|X_n(\omega)| < \infty$ a.s. for $\omega \in [0,1]$. Then there exists a sequence $\{C_n\}$ of positive real numbers such that:$\frac{X_n(\omega)}{c_n} \to 0$ a.s

Given a sequence of random numbers $\{C_n\}$ consider the set $E_n = \{\omega : \frac{|X_n(\omega)|}{c_n} > \frac{1}{n}\}$. Suppose that there is not sequence of positive numbers $\{c_n\}$ such that $Pr(E_n) \leq 2^{-n}$. Then WLOG we can assume that $\{C_n\}$ is a sequence of positive numbers. Fix $n \in \mathbb{N}$. Then it follows that for any $N \in \mathbb{N}$, $Pr(A_{N,}) = P_r(\omega : \frac{|X_n(\omega)|}{N} > \frac{1}{n}) > 2^{-n}$, which is $P_r(\omega : |X_n(\omega)| > \frac{N}{n}) > 2^{-n}$.

So $A_1 = \{\omega : X_n(\omega) > \frac{1}{n}\}$, $A_2 = \{\omega : X_n(\omega) > \frac{2}{n}\}...A_\infty = \{\omega : X_n(\omega) = \infty\}$. Hence the sequence is such that $A_1 \supset A_2 \supset ....$ Then $A_\infty = \cap_{i=i}^{\infty} A_i$. Hence $Pr(\cap_{i=i}^{\infty} A_i) = P_r(A_\infty) > 2^{-n}$, however by assumption$P_r(A_\infty) = 0$, a contradiction.

Hence there exist a sequence s.t. $Pr(E_n) = P_r(\omega : \frac{|X_n(\omega)|}{C_n} > \frac{1}{n}) \leq 2^{-n}$.

Thus the series converges by comparison with the geometric series. $\sum_{n=1}^{\infty} P(E_k) \leq \sum 2^{-n}$, $\sum_{n=1}^{\infty} P(E_k) \leq 1 < \infty$.

And According to Borel-Cantelli, then $Pr\left(\bigcap_{i=1}^{\infty}\bigcup_{j=i}^{\infty} E_j\right) = 0$,

then $\omega \notin \bigcap_{i=1}^{\infty}\bigcup_{j=i}^{\infty} E_j$ then there is a $k_o \in \mathbb{N}$such that $\frac{|X_n(\omega)|}{c_n} < \frac{1}{n}$ for all $n > k_o$, by comparison $\frac{|X_n(\omega)|}{c_n} < \frac{1}{n}$ would converge to zero since $\frac{1}{n}$ converges to zero,so $\frac{X_n(\omega)}{c_n} \to 0 a.s$

2. **(Storng and Weak Law of Large Numbers)**:

(a) Let $\{X_n, n \geq 2\}$be a sequence of independent random variables, such that; $P(X_n = \pm n) = \frac{1}{2nlogn}$, $P(X_n = 0) = 1 - \frac{1}{nlogn}$, $n = 2, 3, ...$

i. Show that the SLLN does not hold

Note that $E[X_n] = 0$,

Consider the events $A_n = \{|X_n| \geq n\}$, with $n \geq 2$. Then $P(A_n) = \frac{1}{nlogn} \implies \sum_{i=2}^{n} P(A_n) = \infty$

hence by the second Borel-Cantelli lema $P\left(|X_n| \geq n, \, i.o\right) = 1$

hence that means that no matter how far you go in the sequence, then $lim_{n\to\infty} P\left(\cup_{i=n}^{\infty} |X_i| \geq n, \right) = 1$, and so

Note that the difference of $\bar{X}_{n+1} - \bar{X}_n$ can be written as $\frac{s_{n+1}}{n+1} - \frac{S_n}{n} = \frac{X_{n+1}}{n+1} - \frac{1}{n(n+1)} \sum X_k$ and since $\sum X_k < \sum k$, then $\frac{s_{n+1}}{n+1} - \frac{S_n}{n} \geq 1 - \frac{n(n+1)}{2n(n+1)} = \frac{1}{2}$, for infinitely many n so $P\left(lim\, sup \bar{X_n} \geq \frac{1}{2}\right) = 1$ hence it cannot converge to 0 a.s.

ii. Show that WLLN of holds

First note that $VarX_k = E[X_k^2] = \frac{k}{logk}$.

Note that $\frac{k}{logk}$ its minimum is at $k = e$, and that $\sum_{k=3}^{n} \frac{k}{logk} \leq \int_3^{n+1} \frac{k}{logk} dk$

then $\frac{1}{n^2}\sum_{k=2}^{n} VX_k \leq \frac{1}{n^2}\left[\frac{2}{log2} + \int_3^{n+1} \frac{x}{logx} dx\right] \leq \frac{2}{n^2 log2} + \frac{(n-2)(n+1)}{n^2 log(n+1)} \to 0$ as $n \to \infty$

Hence the Markov condition for $\{X_n\}$ is satisfied and therefore the sequence obeys the WLLN

(b) Let $\{X_n, n \geq 1\}$ be independent r.v.s where the density of X_n is given by: $f_n(x) = \frac{1}{\sqrt{2}\sigma_n} e^{-\sqrt{2}|x|/\sigma_n}$, with $\sigma_n^2 = \frac{2n^2}{(logn)^2}, n \geq 2$

i. Show that SLLN does not holds

Let $A_n = \{|X_n| \geq n\}$, then $P(A_n) = \frac{2}{\sqrt{2\sigma)n}} \int_n^\infty exp\left(\frac{-\sqrt{2}x}{\sigma_n}\right) dx = exp\left(-\frac{1}{2}\sqrt{2}(logn)^2/n\right)$ since $(logn)^2/n \to 0$ as $n \to \infty$, hence $\sum_{i=2}^\infty P(A_n) = \infty$, by the same logic as before we get that SLLN does not hold

3. **(Point Estimation)** We are interested on giving an estimate of μ and σ^2.

(a) Suppose that we have a i.i.d random sample, $\{X_i\}_{i=1}^n$,are the following estimators are unbiased?

i. $\hat{\mu} = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} X_i$, $\hat{\mu}^* = X_1$ $\hat{\hat{\mu}} = (X_1 - X_2 + X_3 - X_4 + ... + X_n)$

$E[\hat{\mu}]$=$\frac{1}{n}\sum E[X_i] = \mu$, so unbiased

$E[\hat{\mu}^*] = E[X_1] = \mu$ so unbiased

$E[\hat{\hat{\mu}}] = \mu - \mu + \mu - .. + \mu = \mu$

ii. $S^2 = \frac{1}{n}\sum(x_i - \bar{x})^2$ $S^{2*} = \frac{1}{n-2}\sum(x_i - \bar{x})^2$, $\hat{S^2} = \frac{1}{n-1}\sum(x_i - \bar{x})^2$

$E[S^2] = \frac{1}{n}\sum E[(x_i - \bar{x})^2] = \frac{1}{n}\sum E[((x_i - \mu) - (\bar{x} - \mu))^2]$

$\frac{1}{n}\sum E[((x_i - \mu) - (\bar{x} - \mu))^2] = \frac{1}{n}\sum E[(x_i - \mu)^2] - \frac{2}{n}\sum E[(x_i - \mu)(\bar{x} - \mu)] + \frac{1}{n}\sum E[(\bar{x} - \mu)^2]$

or $\sigma - E[\frac{2}{n}(\bar{X} - \mu)n(\bar{X} - \mu)] + E[(\bar{x} - \mu)^2]$

$E[S^2] = \sigma - E[(\bar{X} - \mu)^2]$ so biased similarly $E[S^{2*}]$ is biased

but it is possible to check that $\hat{S^2}$ is unbiased