Econ 704 Discussion Section-Handout 6

Emilio Cuilty,

Handout desingned by: Jian Zhang

cardenascuil@wisc.edu . Office: 7143, Office Hours: Thursdays 9:00-11:00 A.M.

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1. Review:

- Point Estimation: Suppose that $\{X_i : 1 \le i \le n\}$ are a sequence of observations (or trials). An estimator T_n is a random variable that is a **function** of $\{X_i : 1 \le i \le n\}$. Let β be the unknown parameter that is of interest.
  1. T_n is an *unbiased* estimator of β if $E[T_n] = \beta$.
  2. T_n is a *consistent* estimator of β if T_n converges in probability to β.
  3. Suppose that S_n and T_n are two **unbiased** estimators of β. Then T_n is *more efficient* than S_n if $Var(T_n) < Var(S_n)$.

- Interval Estimation: $\{X_i : 1 \le i \le n\}$ is a sequence of i.i.d. random variables with $EX_1 = \mu$ and $Var(X_1) = \sigma^2$. Suppose that μ is the parameter that is of interest. Then its $1-\alpha$ interval estimator can be derived as follow:
  1. If σ is known, then it is $\left[\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$.
  2. If σ is unknown, then just replace σ with its estimator $S_n = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$. Furthermore, if $X_i \sim Bernoulli(p)$, then we use $S_n = \sqrt{\bar{X}(1-\bar{X})}$.

  Remark: The above results are valid only if sample size n is large enough and thus CLT can be applied.

2. Problems

1. **Estimation**.True or False?
   (a) If T_n is an unbiased estimator, then it is also consistent.
   False. Counterexample: Let $\{X_i\}$ be a sequence of i.i.d. random variables with distribution $N(\mu, 1)$. Then $T_n = X_1$ is a unbiased estimator of μ but $T_n = X_1$ doesn't converge to μ in probability
   (b) If T_n has a smaller variance than S_n, then T_n is more efficient.
   False. To compare the efficiency of two estimators, we need that both of them are unbiased
   (c) Brock derives the 95% confidence interval of β, which is $[-1.97, 2, 97]$. Then the probability that $\beta \in [-1.97, 2.97]$ is 0.95.
   False. β is not a random number. Then this probability could either be 0 or 1.
2. **Point Estimation.** Suppose T_n is an estimator of β. The bias of T_n is $\delta = E(T_n) - \beta$ and variance of T_n is $Var(T_n) = \sigma^2$. Show that $E(T_n - \beta)^2 = \delta^2 + \sigma^2$.

$$E(T_n - \beta)^2 = E\big((T_n - E(T_n)) + (E(T_n) - \beta)\big)^2 = E(T_n - E(T_n))^2 + \delta^2 + 2\delta E(T_n - E(T_n)) = \sigma^2 + \delta^2.$$

3. **Point Estimation**. Suppose $\{X_i : 1 \le i \le n\}$ are i.i.d. sequence of random variables. $X_i \sim Uniform(0, b)$, i.e., the CDF of X_i is $F(x) = x/b$ if $0 \le x \le b$. Here, we assume that $b > 0$.

(a) Calculate the CDF and PDF of $\max_{1\le i\le n} X_i$.

Denote its CDF and PDF by F_m and f_m respectively. Then

$$F_m(x) = \Pr(\max_{1\le i\le n} X_i \le x) = \Pr(X_i \le x, i = 1, 2, \cdots, n) = \prod_{i=1}^{n} \Pr(X_i \le x) = \big(F(x)\big)^n.$$

Therefore, $F_m(x) = \left(\frac{x}{b}\right)^n$ if $0 \le x \le b$, $F_m(x) = 1$ if $x > b$ and $F_m(x) = 0$ if $x < 0$. Since $f_m(x) = F_m'(x)$, then

$$f_m(x) = \frac{nx^{n-1}}{b^n} \text{ if } x \in [0, b]; \qquad f_m(x) = 0 \text{ otherwise.}$$

(b) Find a real number a_n only depended on n such that $T_n = a_n \max_{1\le i\le n} X_i$ is an unbiased estimator of b.

$E[T_n] = a_n E[\max_{1\le i\le n} X_i] = a_n \int_{-\infty}^{\infty} x f_m(x)dx = a_n \int_0^b n\frac{x^n}{b^n}dx = a_n \frac{n}{n+1}b$. In order to make T_n an unbiased estimator, we just need that $a_n \frac{n}{n+1} = 1$, that is, $a_n = \frac{n+1}{n}$.

(c) Let $S_n = 2\bar{X}$ be another unbiased estimator of b and T_n is as Part (b). Show that T_n is more efficient than S_n if $n \ge 2$.

Since S_n and T_n are unbiased, we just need to compare their variances.

$$Var(S_n) = 4Var(\bar{X}) = \frac{4}{n}Var(X_1) = \frac{b^2}{3n}.$$

$$Var(T_n) = a_n^2 Var\left(\max_{1\le i\le n} X_i\right) = \left(\frac{n+1}{n}\right)^2 \left[\int_0^b x^2 \frac{nx^{n-1}}{b^n}dx - \left(\frac{n}{n+1}b\right)^2\right] = \frac{b^2}{n(n+2)}.$$

Since $n \ge 2$, then $n + 2 > 3$ and thus $n(n+2) > 3n$. Therefore, $Var(T_n) < Var(S_n)$.

(d) Show that T_n is consistent.

For any $\varepsilon > 0$, by Chebyshev's inequality,

$$\Pr(|T_n - b| \ge \varepsilon) \le \frac{Var(T_n)}{\varepsilon^2} = \frac{1}{n(n+2)}\frac{b^2}{\varepsilon^2}.$$

Sending $n \to \infty$, then $\lim_{n\to\infty} \Pr(|T_n - b| \ge \varepsilon) = 0$.

(e) (Borel-Cantelli) Show that $T_n \overset{a.s.}{\to} b$.

Denote $A_n = \{|T_n - b| \ge \varepsilon\}$. Then by Part (c), it has

$$\Pr(A_n) \le \frac{1}{n(n+2)}\frac{b^2}{\varepsilon^2} < \frac{1}{n^2}\frac{b^2}{\varepsilon^2}.$$

Then $\sum_{n=1}^{\infty} \Pr(A_n) < \infty$. By Borel-Cantelli lemma, $\Pr(A_n \text{ i.o.}) = 0$. Therefore, $T_n \overset{a.s.}{\to} b$.

(f) Show that $n\big(\max_{1\le i\le n} X_i - b\big)$ converges in distribution to a distribution F_b with its CDF $F_b(x) = e^{x/b}$ if $x \le 0$ and $F_b(x) = 1$ if $x > 0$.

If $x > 0$, then $\Pr(n\big(\max_{1\le i\le n} X_i - b\big) \le x) = \Pr(\max_{1\le i\le n} X_i \le b + x/n) = 1$;
If $x \le 0$, then

$$\Pr\left(n\big(\max_{1\le i\le n} X_i - b\big) \le x\right) = \Pr\left(\max_{1\le i\le n} X_i \le b + \frac{x}{n}\right) = \left(\frac{b + x/n}{b}\right)^n = \left(1 + \frac{1}{n}\frac{x}{b}\right)^n \to e^{x/b}.$$

4. **Point Estimation**.Suppose that $\{X_i : 1 \le i \le n\}$ are a i.i.d. sequence of random variables with $X_i \sim Bernoulli(p)$. Let $S_n^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$. Show that
   (a) $S_n^2 = \frac{n}{n-1}\bar{X}(1-\bar{X}) \approx \bar{X}(1-\bar{X})$.

   Note that X_i can only be 0 or 1. Then $X_i^2 = X_i$. Therefore,

$$S_n^2 = \frac{n}{n-1}\frac{1}{n}\left(\sum_{i=1}^n X_i^2 - n\bar{X}^2\right) = \frac{n}{n-1}\frac{1}{n}\left(\sum_{i=1}^n X_i - n\bar{X}^2\right) = \frac{n}{n-1}\left(\bar{X} - (\bar{X})^2\right) = \frac{n}{n-1}\bar{X}(1-\bar{X}).$$

   (b) $E[\bar{X}(1-\bar{X})] = \left(1 - \frac{1}{n}\right)p(1-p)$.

   Recall that S_n^2 is an unbiased estimator of $\sigma^2 = Var(X_1) = p(1-p)$. Therefore,

$$p(1-p) = E(S_n^2) = \frac{n}{n-1}E[\bar{X}(1-\bar{X})].$$

   Therefore, $E[\bar{X}(1-\bar{X})] = \left(1 - \frac{1}{n}\right)p(1-p)$.

5. **Interval Estimation** In 100 tosses of a coin, there are 60 heads and 40 tails. Let p be the probability of head in a toss of this coin. What is the 95% confidence interval of p? Do you believe that p could be 0.5?

   Let $X_i = 1$ if i-th toss of this coin is head, otherwise, $X_i = 0$. Then $X_i \sim bernoulli(p)$. Therefore, the 95% confidence interval of p is,

$$\left[\bar{X} - 1.96\sqrt{\frac{\bar{X}(1-\bar{X})}{n}}, \bar{X} + 1.96\sqrt{\frac{\bar{X}(1-\bar{X})}{n}}\right] = [0.6 - 1.96\frac{\sqrt{24}}{100}, 0.6 + 1.96\frac{\sqrt{24}}{100}] \approx [0.504, 0.696]$$

   Since $0.5 \notin [0.504, 0.696]$, we tend to believe that p is not 0.5.

6. **Interval Estimation** Suppose that $\{X_i : 1 \le i \le n\}$ is a i.i.d. sample from a distribution $exponential(\lambda)$, i.e., its density function is that $f(x) = \lambda e^{-\lambda x}$ if $x > 0$ and $f(x) = 0$ otherwise. Here, we assume that $\lambda > 0$.
   (a) Calculate $E[X_i]$ and $Var(X_i)$.

   $E[X_i] = \int_0^\infty x \cdot \lambda e^{-\lambda x} dx \overset{t=\lambda x}{=} 1/\lambda \int_0^\infty t e^{-t} dt = \lambda$. Moreover,

$$E(X_i^2) = \int_0^\infty x^2 \lambda e^{-\lambda x} dx = -\int_0^\infty x^2 de^{-\lambda x} = -x^2 e^{-\lambda x}\Big|_0^\infty + \int_0^\infty 2x e^{-\lambda x} dx = \frac{2}{\lambda^2}.$$

   Therefore, $Var(X_i) = E[X_i^2] - (EX_i)^2 = \frac{1}{\lambda^2}$.

(b) For large enough n, use CLT to calculate the coverage probability of the following interval:

$$\lambda \in \left[\frac{1}{\bar{X} + \frac{1}{\lambda\sqrt{n}}}, \quad \frac{1}{\bar{X} - \frac{1}{\lambda\sqrt{n}}} \right]$$

Note that

$$\begin{aligned}
\Pr\left(\lambda \in \left[\frac{1}{\bar{X} + \frac{1}{\lambda\sqrt{n}}}, \quad \frac{1}{\bar{X} - \frac{1}{\lambda\sqrt{n}}} \right]\right) &= \Pr\left(\bar{X} - \frac{1}{\lambda\sqrt{n}} \le \frac{1}{\lambda} \le \bar{X} + \frac{1}{\lambda\sqrt{n}}\right) \\
&= \Pr\left(-1 \le \frac{\bar{X} - \lambda^{-1}}{\lambda^{-1}/\sqrt{n}} \le 1 \right) \\
&= \Pr\left(-1 \le \frac{\bar{X} - E(X_i)}{SD(X_i)/\sqrt{n}} \le 1 \right) \\
&\approx \Pr(-1 \le Z \le 1) \approx 0.683.
\end{aligned}$$