Lectures on Advance Macroeconomics

Emilio Cuilty Luna

March 28, 2013

My lectures are written to my students. Please talk to me before circulating the notes. I deeply appreciate all comments and help to improve these lecture notes.

Syllabus

Objective: The main purpose of this course is to review the economic fluctuations, review the ideas from the neo-classic and Neokeynesian schools and finally attempt to give some behavioral explanations. The quantitative mathematical or analytical requirements are very high, some review will be given on relevant tools for solve problems nevertheless advanced calculus, econometrics and dynamic programming will be assumed. However the intuition will be the most important part to develop. The core of the course will be centered in Romer's book; from time to time it will be necessarily to review other texts such as Stokey and Lucas or Summers' papers. At the end of the course the approved student will understand the differences in the explanations of both schools and how to solve standard problems of macroeconomic fluctuations. The student will also be able give clean proofs of the theorems or find the relevant results of the models.

Approve: This course is far from being an easy course, and probably will consume an important part of your time during the semester. My recommendation to approve the course is to study a lot. It is almost impossible to approve just reading the class notes. HARD WORK is required to approve. It is expected that the student will read the relevant chapter of the book at least once before entering the classroom, more readings will be needed to complete homework or to fully understand the theory.

Ethics: I believe that as future economists you should always be guided by social norms and ethic codes. According to the former idea you might NEVER cheat or copy in a homework, exam or whatever. Those students failing to behave honestly will be punished with an automatic D.A. with no consideration to negotiate. My grading scheme is designed to fail anyone that receives a D.A., so please don't try to make tricks on me. It is important to be respectful with others ideas, please be kind with everybody questions and pay attention to all ideas, they might appear on the exams or quizzes.

Grading scheme

- Two Partial exams, 60% of the final grade
- Final evaluation, 30% of the final grade
- 6 or 5 quizzes, 10% of the final grade
- Bonus, extra 10% and will be only available for 2 or 3 students

Exams: The first partial exam will be on February 14, the second partial exam will be on April 4 and the final evaluation will be on May 9. These dates are fixed and I will not move any exam for any reason. So if you have more than one exam this day I suggest to you to try to move the other exam(s), otherwise good luck. Both partial exams cover all the topics we review in class or in the Problem Sets. The second partial exam covers the topics of the second partial and the first one. The final evaluation will comprehend only the second partial and the last topics covered.

Quizzes: There will be 5 or 6 surprise quizzes. The quizzes will be one or two short questions about the material covered last week. The quizzes will be applied at 7:07 with a maximum length of 10 minutes. If you arrive later than 7:17 you will not have access to the quiz and will count as zero. Nevertheless you should come to class even if you miss the quiz, otherwise you will miss important material.

Problem-Sets: There will be 4 or 5 problem sets. The problem sets are complicated and will require a lot of time to complete them, so start them as soon as possible. You might work in groups, pairs or whatever. The problem sets do not account for any points on the final grade, however not completing them will be a dangerous move, in order to master theory you need to work, fail, and rethink. Problem Sets will be answered in a class lecture, however you must present your Problem Set at the beginning of the session otherwise you will be not allowed to enter.

Bonus: I do not yet how to account for bonus, nevertheless I will think about it.

Topics

- Introduction, Where is the State of the Macro today?
  - Krugman, P. "How Did Economists Get It So Wrong?" (2009), New York Times
- Real-Business-Cycle Theory, The freshwater's new hope
  - Romer, D. "Advance Macroeconomics" (2012) McGraw-Hill 4th edition. Chapter 5
  - Mankiw, G. "Macroeconomics" (2000) 4th edition. Chapter 19
  - Stokey, N. and R. Lucas "Recursive Methods in Economics Dynamics" Harvard Press 1th edition. Chapters 4, 5, 6
- Nominal Rigidity, The Neokeynesian strikes back
  - Romer, D. "Advance Macroeconomics" (2012) McGraw-Hill 4th edition. Chapter 6
- Dynamic Stochastic General Equilibrium Models of Fluctuation, The return of the Neoclassic school
  - Romer, D. "Advance Macroeconomics" (2012) McGraw-Hill 4th edition. Chapter 7
  - Mas-Colell,A., M. Whinston and J. Green, "Microeconomic Theory" (1995) Oxford Press 1th edition. Chapters 16 and 19
  - Stokey, N. and R. Lucas "Recursive Methods in Economics Dynamics" Harvard Press 1th edition. Chapters 9 and 10
- Behavioral Finance and Behavioral Macroeconomics. Get rid of that non sense
  - Long, J. B. D., A. Shleifer, L. H. Summers, and R. J. Waldmann (1991): The Survival of Noise Traders in Financial Markets, The Journal of Business, 64(1), pp. 119.

Part I

Introduction

Chapter 1

Where is the state of the Macro today?

1.1 A clever discussion of Macroeconomics.

In September of 2009 the Nobel laureate Paul Krugman published an interesting article in the New York Times, How Did Economists Get It So Wrong?. This first class we are going to review the most important points of the article. Hopefully at the end of the class you will understand better why is important to dig deeper in to macroeconomics (and why society expects from you to complete several courses on macro) even if history has showed us that this science do not work at all. Most of you may not remember but when I was a student, here at ITESM, the academic economic society were congratulating them self on how they had fully understood how macroeconomics work. In fact Oliver Blanchard of M.I.T. declared that the state of the Macro was good, and the eternal fight between neoclassic school and neokeynesian view ended with an almost perfect discipline. Thanks to the "well designed public policy" the most important economy (yes, the United States of America) had experienced a period of bonanza, called the great moderation. Even Mexico, back in 2006, was a promising economy, with a central bank taking care of inflation and a new nice government fighting "economic recessions".

But as Krugman mentions, in 2008 everything came apart.

The economic-financial-debt-or whatever crisis of 2008 was not predicted by almost anyone, even if old Ben Bernanke was on guard 24 7. However Krugman argues that this is not the biggest problem. The big issue with the field of macroeconomics is the blindness of the profession, the dogma in the great market and the nice touch of the government. Krugman believes that the economics profession went astray because economists, as a group, mistook beauty, clad in impressive-looking mathematics, for truth.

I believe that most sciences work when there is no problem surrounding them. Nevertheless when the society faces an euro zone in crisis and America with almost a zero interest rate!(leaving no room to monetary policy) the scientific community search for a change (a quick, cheap and beautiful change). In the words of Krugman "Unfortunately, this romanticized and sanitized vision of the economy led most economists to ignore all the things that can go wrong. They turned a blind eye to the limitations of human rationality that often lead to bubbles and busts; to the problems of institutions that run amok; to the imperfections of markets — especially financial markets — that can cause the economy's operating system to undergo sudden, unpredictable crashes; and to the dangers created when regulators don't believe in regulation".

1.1.1 From Smith to Keynes and Back

I do believe that you believe as society believes that economics was created by Adam Smith, in the 1776 famous book The Wealth of the Nations. The central (macro) message of the Wealth of the Nations and the next 160 years of research was TRUST THE MARKET!. In 1844 was published the novel by Alexander Dumas the count of Monte Cristo. You might remember that the count decided to study economics to become a successful man, because back there those whom studied economics (i.e. The Wealth of the Nations) will understand better the market and could take advantages from it. Nevertheless in 1929 something went wrong with the market, destroying everything our old friend from Monte Cristo believed. Some years later in the book of Macroeconomics of Parking, the author claims that economist are the clever guys in the world because they know how to live with limited resources and to be happy with them, we haven't learn anything.

According to Krugman besides what you have heard, Keynes did not wanted to live in world with a government taking care of everybody. He described his analysis in his 1936 masterwork, "The General Theory of Employment, Interest and Money," as "moderately conservative in its implications." He wanted to fix capitalism, not replace it. The truth about the general theory is the fact that government can, and should make intervention when some shock or depression haunts the nation. Here at the ITESM you have previously review some important implications about Keynes. Most of them are found in the book of Mankiw.

Then Milton Friedman arrived, with the new monetarism doctrine. Monetarists did not disagree with the idea that government should make interventions, in fact Friedman once sated "We are all Keynesians now", however the government should only take care of eliminate externalities, instructing central banks to keep and the nation's money supply, the sum of cash in circulation and bank deposits, growing on a steady path. There is a great great book that you should read whenever you have some extra time called Capitalism and Freedoms by Friedman.

Eventually, however, the anti-Keynesian counterrevolution went far beyond Friedman's position, which came to seem relatively moderate compared with what his successors were saying. Among financial economists, Keynes's disparaging vision of financial markets as a "casino" was replaced by "efficient market" theory, which asserted that financial markets always get asset prices right given the available information (this means that no one will ever pay more than the efficient price for an action or a bound, in other words those LAFS and LEFs from ITESM will always charge the optimum for every stock, bond or whatever)

Not all macroeconomists were willing to go down this road: many became self-described New Keynesians, who continued to believe in an active role for the government. Yet even they mostly accepted the notion that investors and consumers are rational and that markets generally get it right (Krugman is one of those). Not everyone was a Keynesian or a neoclassic, there were a few that believed that humans were not perfect machines, and that rationality might not hold in the financial markets, but they were fighting against the tide. If you want to know more about them, you should read papers of Schlifer,Labison, and some of Summers.

1.1.2 Panglossian Finance

According to Keynes the financial markets can be understood as beauty contests. In order to understand them you must not pick your favorite girl, but the one you believe will be the favorite of the jury. Nevertheless in the 70's the financial world was understood as perfect, we humans have nothing to worry about bubbles, investors irrationality and destructive speculation. Fama's hypothesis dominated the theory, this theory claims that the price you see on an stock is exactly the true value of the company given the information of the company's earnings. According to Krugman some Harvard economists believe in Fama so blind that their recommendation was to maximize the stock prices of a company for a better economy.

I am pretty sure you remember the CAPM model. It is a beautiful model with very nice implications, how to value an asset and how to put a price to almost every derivative. But it only works in the scheme of efficient markets, so in my opinion does not work. Krugman argues that financial economist rarely asked themselves if the price actually reflected the world fundamentals. Instead they asked if the asset prices reflected the other asset prices. Summers viewed financial economist as ketchup economist, if a two quarts bottle sell invariable for exactly as twice as much of a quart bottle of ketchup then the market is efficient.

According to Krugman with the mocks and everything, nothing change. At that time Greenspan who was then the Fed chairman and a long-time supporter of financial deregulation whose rejection of calls to rein in sub prime lending or address the ever-inflating housing bubble rested in large part on the belief that modern financial economics had everything under control. By October of last year, however, Greenspan was admitting that he was in a state of "shocked disbelief," because "the whole intellectual edifice" had "collapsed."What should policy makers do? Unfortunately, macroeconomics, which should have been providing clear guidance about how to address the slumping economy, was in its own state of disarray

1.1.3 The Trouble with Macro

Consider the travails of the Capitol Hill Baby-Sitting Co-op. This co-op, whose problems were recounted in a 1977 article in The Journal of Money, Credit and Banking, was an association of about 150 young couples who agreed to help one another by baby-sitting for one another's children when parents wanted a night out. To ensure that every couple did its fair share of baby-sitting, the co-op introduced a form of scrip: coupons made out of heavy pieces of paper, each entitling the bearer to one half-hour of sitting time. Initially, members received 20 coupons on joining and were required to return the same amount on departing the group.

Unfortunately, it turned out that the co-op's members, on average, wanted to hold a reserve of more than 20 coupons, perhaps, in case they should want to go out several times in a row. As a result, relatively few people wanted to spend their scrip and go out, while many wanted to baby-sit so they could add to their hoard. But since baby-sitting opportunities arise only when someone goes out for the night, this meant that baby-sitting jobs were hard to find, which made members of the co-op even more reluctant to go out, making baby-sitting jobs even scarcer. . . .In short, the co-op fell into a recession

The question is whether this particular example, in which a recession is a problem of inadequate demand — there isn't enough demand for baby-sitting to provide jobs for everyone who wants one — gets at the essence of what happens in a recession.

Forty years ago most economists would have agreed with this interpretation. But since then macroeconomics has divided into two great factions: "saltwater" economists (mainly in coastal U.S. universities), who have a more or less Keynesian vision of what recessions are all about; and "freshwater" economists (mainly at inland schools), who consider that vision nonsense.

Freshwater economist are those who does not believe in this explanation. The central fact is that markets work, so there cannot be a lack of demand, because prices always move to match supply and demand. If people want more coupons the price will rise. So why we see that there is unemployment in recessions?

In the 1970s the leading freshwater macroeconomist, the Nobel laureate Robert Lucas, argued that recessions were caused by temporary confusion: workers and companies had trouble distinguishing overall changes in the level of prices because of inflation or deflation from changes in their own particular business situation. And Lucas warned that any attempt to fight the business cycle would be counterproductive: activist policies, he argued, would just add to the confusion.

Edward Prescott, who was then at the University of Minnesota, argued that price fluctuations and changes in demand actually had nothing to do with the business cycle. Rather, the business cycle reflects fluctuations in the rate of technological progress, which are amplified by the rational response of workers, who voluntarily work more when the environment is favorable and less when it's unfavorable. Unemployment is a deliberate decision by workers to take time off.

But the basic premise of Prescott's "real business cycle" theory was embedded in ingeniously constructed mathematical models, which were mapped onto real data using sophisticated statistical techniques, and the theory came to dominate the teaching of macroeconomics in many university departments. In 2004, reflecting the theory's influence, Prescott shared a Nobel with Finn Kydland of Carnegie Mellon University.

On the other side Saltwater economists believe that the demand side explanation was too compelling to reject. N. Gregory Mankiw at Harvard, Olivier Blanchard at M.I.T. and David Romer at the University of California, Berkeley and of course Paul Krugman of M.I.T. were trying to add enough imperfections to accommodate a more or less Keynesian view of recessions, so active policy to fight recessions remained desirable.

But the self-described New Keynesian economists weren't immune to the charms of rational individuals and perfect markets. They tried to keep their deviations from neoclassical orthodoxy as limited as possible. This meant that there was no room in the prevailing models for such things as bubbles and banking-system collapse.

Nevertheless around 1985 and 2007 the fight was only about theory and not about policy making. New Keynesians, unlike the original Keynesians, didn't think fiscal policy — changes in government spending or taxes — was needed to fight recessions. They believed that monetary policy, administered by the technocrats at the Fed, could provide whatever remedies the economy needed. At a 90th birthday celebration for Milton Friedman, Ben Bernanke, formerly a more or less New Keynesian professor at Princeton, and by then a member of the Fed's governing board, declared of the Great Depression: "You're right. We did it. We're very sorry. But thanks to you, it won't happen again." The clear message was that all you need to avoid depressions is a smarter Fed. freshwater economists found little to complain about. (They didn't believe that monetary policy did any good, but they didn't believe it did any harm, either.)

1.1.4 Nobody could have predicted...

It's what you say with regard to disasters that could have been predicted, should have been predicted and actually were predicted by a few economists who were scoffed at for their pains. Robert Shiller did a prediction of the bubble the house market. But policy maker did not see anything, Carstens among them. Home price increases, Ben Bernanke said in 2005, "largely reflect strong economic fundamentals." Agustin Carstens limited to say that Mexico will suffer only from a Catarrito.

Eugene Fama, the father of the efficient-market hypothesis, declared that "the word 'bubble' drives me nuts," and went on to explain why we can trust the housing market: "Housing markets are less liquid, but people are very careful when they buy houses. It's typically the biggest investment they're going to make, so they look around very carefully and they compare prices. The bidding process is very detailed." Yes people compared prices, but among other bad prices, just ketchup economics again.

U.S. households have seen $13 trillion in wealth evaporate. More than six million jobs have been lost, and the unemployment rate appears headed for its highest level since 1940. So what guidance does modern economics have to offer in our current predicament? And should we trust it?

1.1.5 The Stimulus Squabble

the crisis ended the phony peace. Suddenly the narrow, technocratic policies both sides were willing to accept were no longer sufficient — and the need for a broader policy response brought the old conflicts out into the open, fiercer than ever.

The Fed dealt with the recession that began in 1990 by driving short-term interest rates from 9 percent down to 3 percent. It dealt with the recession that began in 2001 by driving rates from 6.5 percent to 1 percent. And it tried to deal with the current recession by driving rates down from 5.25 percent to zero. But zero, it turned out, isn't low enough to end this recession. And the Fed can't push rates below zero, since at near-zero rates investors simply hoard cash rather than lending it out.

Fiscal stimulus is the Keynesian answer to the kind of depression-type economic situation we're currently in. Such Keynesian thinking underlies the Obama administration's economic policies and the freshwater economists are furious.Chicago's Cochrane, outraged at the idea that government spending could mitigate the latest recession, declared: "It's not part of what anybody has taught graduate students since the 1960s. They [Keynesian ideas] are fairy tales that have been proved false. It is very comforting in times of stress to go back to the fairy tales we heard as children, but it doesn't make them less false." (It's a mark of how deep the division between saltwater and freshwater runs that Cochrane doesn't believe that "anybody" teaches ideas that are, in fact, taught in places like Princeton, M.I.T. and Harvard.)

Yet if the crisis has pushed freshwater economists into absurdity, it has also created a lot of soul-searching among saltwater economists. Their framework, unlike that of the Chicago School, both allows for the possibility of involuntary unemployment and considers it a bad thing. But the New Keynesian models that have come to dominate teaching and research assume that people are perfectly rational and financial markets are perfectly efficient. To get anything like the current slump into their models, New Keynesians are forced to introduce some kind of fudge factor that for reasons unspecified temporarily depresses private spending. (I've done exactly that in some of my own work.) And if the analysis of where we are now rests on this fudge factor, how much confidence can we have in the models' predictions about where we are going?

The state of macro, in short, is not good. So where does the profession go from here?

1.1.6 Flaws and Frictions

According to Krugman this should be the future of finance, and I do believe it too.

The good news is that we don't have to start from scratch. Even during the heyday of perfect-market economics, there was a lot of work done on the ways in which the real economy deviated from the theoretical ideal. There is work on behavioral finance. First, many real-world investors bear little resemblance to the cool calculators of efficient-market theory: they're all too subject to herd behavior, to bouts of irrational exuberance and unwarranted panic. Second, even those who try to base their decisions on cool calculation often find that they can't, that problems of trust, credibility and limited collateral force them to run with the herd.

Larry Summers once began a paper on finance by declaring: "THERE ARE IDIOTS. Look around." But what kind of idiots (the preferred term in the academic literature, actually, is "noise traders") are we talking about? Behavioral finance, drawing on the broader movement known as behavioral economics, tries to answer that question by relating the apparent irrationality of investors to known biases in human cognition, like the tendency to care more about small losses than small gains or the tendency to extrapolate too readily from small samples (e.g., assuming that because home prices rose in the past few years, they'll keep on rising).

1.1.7 Re-Embracing Keynes

Here are the last recommendations from Krugman, "First admit that financial markets fall far short of perfection, Second, they have to admit — and this will be very hard for the people who giggled and whispered over Keynes — that Keynesian economics remains the best framework we have for making sense of recessions and depressions. Third, they'll have to do their best to incorporate the realities of finance into macroeconomics."

1.1.8 So why study macro and old ideas?

I have to admit that I am not fully convinced with the vision of Krugman, nor I am Neokeynesian or a Neoclassic. Nevertheless I do believe that in order to save the Macro we must understand what is wrong with it. To understand what is wrong with the macro we need to master its foundations and try to attack them. I believe society has hope on young economist to came with a solution, there is plenty work to do. In contrast with microeconomics, macroeconomics needs young people to recreate the field. This course will be a review of the models that explain economic fluctuations according to both schools neoclassic and Neokeynesian. Maybe if we have time we will review something about behavioral finance at the intuition, Welcome to Advance Macroeconomics II, all resemble with reality is merely coincidence, really coincidence.

Chapter 2

Real-Business-Cycle Theory, The freshwater's new hope

2.1 A Beginners Model

Assumptions

- The economy is formed by identical consumers (households) and a number of identical firms. Everyone is price taker, so we are under perfect competition
- Consumers live forever
- There are only 3 production inputs: capital K, Labor L and Technology A

Production

The production in the economy is described by a Cobb-Douglas function

$$Y_t = K_t^{\alpha}(L_t A_t)^{1-\alpha}$$

where $0 < \alpha < 1$

All the production is divided in Consumption C, Investment I and Government Purchases G

$$Y_t = G_t + C_t + I_t$$

Each period the capital is depreciated by a constant fraction of δ

$$K_{t+1} = K_t + I_t - \delta K_t$$

Or in other words

$$K_{t+1} = K_t + Y_t - G_t - C_t - \delta K_t$$

Wage as usual is equal to the marginal labor productivity $\frac{\partial Y_t}{\partial L_t} = (1-\alpha)K_t^{\alpha}(L_t A_t)^{-\alpha} A_t = (1-\alpha)\left(\frac{K_t}{L_t}\right)^{\alpha} A_t^{1-\alpha} = w$. Equally the capital is paid by the marginal productivity. $\frac{\partial Y_t}{\partial K_t} = \alpha\left(\frac{L_t A_t}{K_t}\right)^{1-\alpha} = r_t + \delta$

Household

The representative household maximize the expected value of

$$U = \sum_{t=0}^{\infty} e^{-\rho t} u(c_t, 1 - l_t)\frac{N_t}{H}$$

Most of the times when we are talking about discrete time it is expected to use for discount $\frac{1}{(1+\rho)^t}$, nevertheless the model is log-linear so it will be useful to use an exponential discount.

$u(.)$is the instant utility function and ρ is the discount rate. N_t is the population and H is the number of households. Then by the first assumption it follows that $\frac{N_t}{H}$ is the number of members of the household.

We will assume that the populations grows to a constant rate n and for stability reasons $n < \delta$

$$LnN_t = \bar{N} + n_t$$

Or in other words $N_t = e^{\overline{N}+n_t}$ where $\overline{N}$ is the initial level of the population

As you can see the utility function of the households depends on c the consumption per member and $1 - l_t$ the leisure per member. Given the first assumption $c = \frac{C}{N}$ and $l = \frac{L}{N}$. For simplicity we will additional assume a utility function log-linear

$$u_t = Lnc_t + bln(1 - l_t)$$

where $b > 0$

Technology and Government

In absence of shocks the technology will grow as $LnA_t = \overline{A_t} + g_t$ where g_T is constant. Nevertheless we will introduce a stochastic shock then $LnA_t = \overline{A_t} + g_t + \widetilde{A_t}$ where $\widetilde{A_t}$ is an auto-regressive process of order 1. $\widetilde{A_t} = \rho_A \widetilde{A_{t-1}} + \epsilon_{A,t}$. Where $\epsilon_{A,t}$ is a white noise and $-1 < \rho_a < 1$.

The government purchases are paid by lump-sum taxes, and they can be described as follow:

$LnG_t = \bar{G} + (n+g)t + \widetilde{G_t}$ where $\widetilde{G_t} = \rho_G \widetilde{G_{t-1}} + \epsilon_{G,t}$. Where $\epsilon_{G,t}$ is a white noise not correlated with $\epsilon_{A,t}$ and -1$<\rho_G<$1, this concludes the model.

Review

Definition 1. Stochastic Process

Given $(\Omega, \Gamma, \varrho)$ probability space a Stochastic Process is an index process of random variables $\{x_t\}_{t\in\mathbb{Z}}$ where τ is the set of indexes

a

Definition 2. White Noise

Given $(\Omega, \Gamma, \varrho)$ probability space, a succession of $\{\epsilon_t\}_{t\in\mathbb{Z}}$ is a white noise if and only if

1. $\forall t, s \; cov(\epsilon_t, \epsilon_s) = 0$ and $s \neq t$
2. $\forall t \; E(\epsilon_t) = 0$
3. $\forall t \; var(\epsilon_t) = \sigma^2 < \infty$

2.2 The households behavior

One Period

Lets start assuming the household only lives one period and does not posses an initial wealth. For simplicity assume only one member per household. The problem of the household then is:

$$Max$$

$$U = Lnc + bLn(1 - l)$$

$$s.a. \; c = wl$$

Using Lagrange method $\mathcal{L} = Lnc + bLn(1-l) + \lambda(wl - c)$

F.O.C.

- $\frac{\partial \mathcal{L}}{\partial c} = \frac{1}{c} - \lambda = 0$
- $\frac{\partial \mathcal{L}}{\partial l} = \lambda w - \frac{b}{1-l} = 0$
- $\frac{\partial \mathcal{L}}{\partial \lambda} = (wl - c) = 0$

The last condition implies that

$$c = wl$$

then by the first we have that

$$\lambda = \frac{1}{wl}$$

and substituting in the second we have that

$$\frac{1}{l} = \frac{b}{1-l}$$

$$\frac{1}{l} = b + 1$$

$$l = \frac{1}{b+1}$$

Observe two things, the first one is that when b is zero the individual decides to work all the time. Nevertheless by the form of the utility function this will be an explosive solution and the indirect utility will be $-\infty$. b measures how much the individual values the leisure over work. The second thing is that in the labor supply and leisure demand the wage has no impact. The intuition behind this idea is that the income effect and the substitution effect are opposite and they cancel each other.

Two Periods No Uncertainty

Assume two periods and no uncertainty on the interest rate or the wage in the second period. The new restriction to the household is

$$c_1 + \frac{1}{1+r}c_2 = w_1 l_1 + \frac{1}{1+r}w_2 l_2$$

By the Lagrange method we have that

$$\mathcal{L} = Lnc_1 + bLn(1-l_1) + e^{-\rho}[Lnc_2 + bLn(1-l_2)] + \lambda[w_1 l_1 + \tfrac{1}{1+r}w_2 l_2 - c_1 - \tfrac{1}{1+r}c_2]$$

Taking the first order conditions of $l's$

- $\frac{\partial \mathcal{L}}{\partial l_1} = \lambda w_1 - \frac{b}{1-l_1} = 0$
- $\frac{\partial \mathcal{L}}{\partial l_2} = \frac{\lambda w_2}{1+r} - e^{-\rho}\frac{b}{1-l_2} = 0$

We could derive the other first order conditions to obtain the demand functions, both to save some time just multiply the first one by $\frac{1}{w_1}$ and the second one by $\frac{1+r}{w_2}$. we have then that $\lambda = \frac{b}{1-l_1}$

2.3 Introduction to Dynamic Programming

Perhaps the easiest way to introduce Dynamic Programming is like Nancy Stokey and Robert Lucas do, giving a concrete example. So lets consider a simple growth model, here we are going to set aside the intuition because we are only interested in the technique and not in the results.

The growth model, the deterministic scenario

Assumptions

A finite number of Identical Households that live forever

Each period only a good is produced y_t with two inputs capital k_t and labor l_t

The production function is $y_t = F(k_t, l_t)$ and is divided into consumption and savings

$$c_t + i_t \leq y_t = F(k_t, l_t)$$

$F(k_t, l_t)$ is continuously differentiable, strictly increasing, strictly quasi concave and homogeneous of degree 1 with

- $F(0, l) = 0$
- $F_k(k, n) > 0$
- $F_n(k, n) > 0$
- For all $k, n > 0$
  - $lim_{k\to\infty} F_k(k, 1) = 0$ and $lim_{k\to 0} F_k(k, 1) = \infty$

The only decision the household has to make is how much to consume and save. The capital depreciates to a constant rate of $-1 < \delta < 1$

$$k_{t+1} = (1-\delta)k_t + i_t$$

Finally the preferences over consumption have the form of

$$\sum_{t=0}^{\infty} \beta^t U(c_t) \quad 0 < \beta < 1$$

Assume also that the population is constant over time and normalize the labor force to 1. The labor supply must satisfy $0 \leq l \leq 1$ all t

Rewriting (for reducing the number of variables) the first restriction as

$$c_t + k_{t+1} - (1-\delta)k_t \leq F(k_t, l_t) \quad all\ t \ \ (1)$$

The households have identical preferences over time and they have an additive form

$$u(c_0, c_1, ...) = \sum_{t=0}^{\infty} \beta^t U(c_t) \quad (2)$$

The utility of the period t $U : \mathbb{R}_+ \to \mathbb{R}$ is bounded, continuously differentiable, strictly increasing and strictly concave. $Lim_{c\to 0} U'(c) = \infty$. Households do not value leisure.

2.3.1 The Problem in Sequence Form

Consider the problem of the social planer which objective is to maximize (2), this is achieved by choosing sequences $\{(c_t, k_t, l_t)\}_{t=0}^{\infty}$ constrained by feasible restrictions given $k_0 > 0$

Observe that in the optimum the product will not be wasted. This means that 1 can be written with equality. We can simplify c_t by substituting the former idea on the utility function. Given the fact that leisure is not valued and that $F_l(k_t, l_t) > 0$ is obvious that $l = 1$ .Then we can say that k_t and y_t are the production and capital per worker, but also the total production and the total capital. It is convenient to define

$$f(k) = F(k, 1) + (1-\delta)k \quad (3)$$

As the total supply of goods in the economy, including the not depreciated capital in the current period

The social planner problem is then

$$\underset{\{(k_{t+1})\}_{t=0}^{\infty}}{Max} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1})$$

$$s.t. \quad 0 \leq k_{t+1} \leq f(k_t) \;\; t = 0, 1, ...$$

$$k_0 > 0$$

It will be useful to start this example with the finite horizon case

$$\underset{\{(k_{t+1})\}_{t=0}^{T}}{Max} \sum_{t=0}^{T} \beta^t U(f(k_t) - k_{t+1})$$

$$s.t. \quad 0 \leq k_{t+1} \leq f(k_t) \;\; t = 0, 1, ...T \quad (4)$$

$$k_0 > 0$$

The sequences $\{(k_{t+1})\}_{t=0}^{T}$ are a close set, bounded and a convex set of $\mathbb{R}^{T+1}$. Additionally note that the objective function is continuous and strictly concave. By the Kuhn-Tucker uniqueness conditions the former idea implies that the problem has an unique solution. To obtain the conditions observe that $f(0) = 0$ and that $U'(0) = \infty$, then the (4) set of restrictions do not bind except for k_{T+1}. It must be true that $k_{T+1} = 0$

Then the solutions must satisfy

$$\beta f'(k_t) U[f(k_t) - k_{t+1}] = U'[f(k_{t-1}) - k_t] \quad (5)$$

$$k_{T+1} = 0 \quad (6)$$

$$k_0 > 0 \;\; given$$

Example 3. Let $f(k) = k^{\alpha}$, $0 < \alpha < 1$ and $U(c) = ln(c)$. Write down equation 5, then apply a change of varibale $z_t = \frac{k_t}{k_{t-1}^{\alpha}}$ to turn the result in a difference equation of order 1 in z_t

We know that $U(c_t) = ln(c_t)$, then $U'(c_t) = \frac{1}{c_t}$. Substituting what we know about consumption$U'(f(k_t) - k_{t+1}) = \frac{1}{k_t^{\alpha} - k_{t+1}}$. Then it must be true that equation five is

$$\beta \alpha k_t^{\alpha - 1} \frac{1}{k_t^{\alpha} - k_{t+1}} = \frac{1}{k_{t-1}^{\alpha} - k_T}$$

Rearranging the terms we have $k_t^{\alpha} - k_{t+1} = \alpha\beta(k_{t-1}^{\alpha} - k_t) k_t^{\alpha - 1}$, then divide everything by k_t^{α}

$$1 - \frac{k_{t+1}}{k_t} = \alpha\beta(k_{t-1}^{\alpha} - k_t) k_t^{-1}$$

or

$$1 - z_{t+1} = \alpha\beta(\frac{1}{z_t} - 1)$$

Finally

$$z_{t+1} = 1 - \alpha\beta(\frac{1}{z_t} - 1)$$

Example 4. Imagine that z_t has reached the stable state (this is only true in the long long run), find the stable(s) state(s)

It must be true that $z_t = z_{t+1} = z$ in the steady state, so from the last equation we have that

$$z = 1 + \alpha\beta - \frac{\alpha\beta}{z}$$

Then solving for z

$$z^2 - (1+\alpha\beta)z + \alpha\beta = 0$$

By the general formula we have that $z_1 = 1$ and $z_2 = \alpha\beta$

Example 5. There must be only one solution to the problem, in fact the condition (6) implies that $k_{T+1} = 0$ which means $z_{T+1} = \frac{0}{k_t^\alpha} = 0$. Find the solution to the difference equation recursively from $z_{T+1} = 0$,

We know that $z_{t+1} = 1 - \alpha\beta(\frac{1}{z_t} - 1)$, solving for z_t

$$\frac{z_{t+1} - 1}{\alpha\beta} = 1 - \frac{1}{z_t}$$

$$z_t = \frac{\alpha\beta}{\alpha\beta - z_{t+1} + 1}$$

Given the fact that $z_{T+1} = 0$ in time T, then

$$z_T = \frac{\alpha\beta}{\alpha\beta + 1}$$

Now if we take the problem to the time $T-1$.

$$z_{T-1} = \frac{\alpha\beta}{(\alpha\beta - z_t + 1)} = \frac{\alpha\beta}{(\alpha\beta - \frac{\alpha\beta}{\alpha\beta+1} + 1)} = \frac{\alpha\beta(\alpha\beta+1)}{\alpha\beta^2 + \alpha\beta - \alpha\beta + \alpha\beta + 1}$$

$$z_{T-1} = \frac{\alpha\beta(\alpha\beta+1)}{(\alpha\beta)^2 + \alpha\beta + 1}$$

Now lets find $T-2$

$$z_{T-2} = \frac{\alpha\beta}{(\alpha\beta - z_{t-1} + 1)} = \frac{\alpha\beta}{(\alpha\beta - \frac{\alpha\beta(\alpha\beta+1)}{(\alpha\beta)^2+\alpha\beta+1} + 1)} = \frac{\alpha\beta((\alpha\beta)^2 + \alpha\beta + 1)}{((\alpha\beta)^3 + (\alpha\beta)^2 + \alpha\beta + 1)}$$

In general for time $T-i$

$$Z_{T-i} = \frac{\alpha\beta(1 + \alpha\beta + ... + (\alpha\beta)^i)}{(1 + \alpha\beta + ... + (\alpha\beta)^i + (\alpha\beta)^{i+1})}$$

And if we say that $i = T - t$

$$z_{T-T+t} = z_t = \frac{\alpha\beta(1 + \alpha\beta + ... + (\alpha\beta)^{T-t})}{(1 + \alpha\beta + ... + (\alpha\beta)^i + (\alpha\beta)^{T-t+1})}$$

To simplify this expression lets say $S \equiv (1+\alpha\beta+...+(\alpha\beta)^{T-t})$ then it must be that $\alpha\beta S = (\alpha\beta+...+(\alpha\beta)^{T-t+1})$. $S - \alpha\beta S = (1 - (\alpha\beta)^{T-t+1})$

$$S = \frac{(1 - (\alpha\beta)^{T-t+1})}{1 - \alpha\beta}$$

Then

$$z_t = \frac{\alpha\beta(\frac{(1-(\alpha\beta)^{T-t+1})}{1-\alpha\beta})}{\frac{(1-(\alpha\beta)^{T-t+2})}{1-\alpha\beta}} = \frac{\alpha\beta(1 - (\alpha\beta)^{T-t+1})}{(1 - \alpha\beta^{T-t+2})}$$

And this is the solution to the problem with z_t, nevertheless we are interested in k_{t+1}. To go back to k_{t+1}, we use the fact that $z_t = \frac{k_t}{(k_{t-1})^\alpha}$ Then

$$\frac{k_t}{k_{t-1}^\alpha} = \frac{\alpha\beta(1-(\alpha\beta)^{T-t+1})}{(1-\alpha\beta^{T-t+2})}$$

or $k_t = \frac{\alpha\beta(1-(\alpha\beta)^{T-t+1})}{(1-\alpha\beta^{T-t+2})}(k_{t-1})^\alpha$, or in $t+1$

$$k_{t+1} = \frac{\alpha\beta(1-(\alpha\beta)^{T-t})}{(1-\alpha\beta^{T-t+1})}k_t{}^\alpha \quad t = 0, 1, ..T$$

The former equation express the dynamic of the capital with reference to the capital of the last period. As you can see future capital is k^α times a constant. However this is only for the finite horizon time. In this class we are interested in the infinite horizon scenario. One can simply guess that the solution to the infinite horizon time is to take the limit when T goes to∞ . $lim_{T\to\infty}k_{t+1} = \frac{\alpha\beta(1-0)}{(1-0)}k_t^\alpha = \alpha\beta k_t^\alpha$

$$k_{t+1} = \alpha\beta k_t^\alpha \quad (8)$$

However is this true?, the answer to this example is yes. Nevertheless we can't assure that for the general case. To proof that the solution to the problem on the infinite horizon is the limit of the finite case a previous course on real analysis and measure theory is required. It is far away from the intentions of this introduction. Then the question is how to solve problems with infinite horizon of time?. The answer was given by Richard Bellman, and in the next part we are going to try to think how Bellman solved the problem

2.3.2 The Bellman Equation

Given what we had learned from (8), we can think that the problem must have a solution $k_{t+1} = g(k_t)$ for $t = 0, 1, ...$ where $g(.)$ is a function of fixed savings. However how does $g(k_t)$ look like?, in fact we have no idea. This motivated Bellman to look for a different approach to the problem. We know that in order to solve the problem of sequences the social planner must satisfy equation (5) and (6), well forget that!. Even that we begin the analysis by choosing sequences of $\{(c_t, k_{t+1})\}_{t=0}^\infty$, the social planner can choose only the consumption today and the capital of tomorrow, the rest of the sequences can wait. This means that the social planer must maximize $c_o$and k_1 over the set of available goods in the economy.

Bellman started thinking that we already know the solution for the problem surrounding equation (3) and (4) for every value of k_0. Let $V : \mathbb{R} \to \mathbb{R}_+$ a function. $V(k_0)$ is the maximum value of (3) for every value of $k_0 \geq 0$. V is known as value function, because it gives the value of the utility maximized (think of V as the indirect utility function of microeconomic theory). So $V(k_1)$ is the maximum value of the utility in period 1. $\beta V(k_1)$ is the maximum value of the utility function of period one but seen from period 0, this means the present value of $V(k_1)$.

It must be true that the problem of the social planer is

$$max_{c_0,k_1}[U(c_0) + \beta V(k_1)] \quad (10)$$

$$s.t.$$

$$c_0 + k_1 \leq f(k_0)$$

$$c_0, k_1 \geq 0 \;\; k_0 > 0 \; given$$

If V is known then we could define $g : \mathbb{R}_+ \to \mathbb{R}_+$ as follows: for every $k_0 \geq 0$ let $k_1 = g(k_0)$ and $c_0 = f(k_0) - g(k_0)$ the values that are on the maximum of (10). Then $g(.)$ completely describes the dynamic of the capital from an initial stock k_0. $g(.)$ is known as the optimal policy function.

Then if solving (10) gives the solution to the social planer, $V(k_0)$ must be the maximized objective function.Then

$$V(k_0) = max_{0 \leq k_1 \leq f(k_0)} \{U[f(k_0) - k_1] + \beta V(k_1)\}$$

Note that the time indexes do not mater in this approach, then

$$V(k) = max_{0 \leq g(k) \leq f(k)} \{U[f(k) - g(k)] + \beta V[g(k)]\}$$

The former equation is known as the Bellman Equation or Functional Equation. The study of optimization problems through functional equations is known as Dynamic Programming. If we assume that V is differentiable and $g(k)$ is interior the first order conditions (11) and envelope conditions (12) are:

$$U'[f(k) - g(k)] = \beta V'[g(k)] \quad (11)$$

$$V'(k) = f'(k) U'[f(k) - g(k)] \quad (12)$$

Example 6. We conjectured that the path for capital given by (8) was optimal for the infinite horizon planning problem, for the functional forms of example 3. Use this conjecture to calculate V by evaluating (2) along the consumption path associated with the capital given by (8).

First lets find equation V, we know that $V(k_0)$ is the maximized value of (2), this means equation (2) valued on c optimum or in $f(k) - g(k)$ optimum. We know from (8) that $g(k) = \alpha\beta k^\alpha$

$$V(k_0) = \sum_{t=0}^{\infty} \beta^t log(k_t^\alpha - \alpha\beta k_t^\alpha)$$

This equation can be rewritten as

$$V(k_0) = \sum_{t=0}^{\infty} \beta^t log(k_t^\alpha (1 - \alpha\beta)) = \sum_{t=0}^{\infty} [\beta^t log(k_t^\alpha) + \beta^t log((1 - \alpha\beta))]$$

Or

$$V(k_0) = \sum_{t=0}^{\infty} [\beta^t \alpha log(k_t)] + \frac{log((1 - \alpha\beta))}{1 - \beta}$$

Next lets look $log(k_t)$.

First of all we need the optimum value for k_t, for that we use the fact that $k_{t+1} = \alpha\beta k_t^\alpha$ To solve the difference equation we iterate given the fact that k_0 if given

In t=1 we have that

$$k_1 = \alpha\beta k_0^\alpha$$

In t=2

$k_2 = \alpha\beta k_1^\alpha = \alpha\beta(\alpha\beta k_0^\alpha)^\alpha = (\alpha\beta)^{\alpha+1} k_0^{\alpha^2}$

Then in t=3

$$k_3 = \alpha\beta k_2^\alpha = \alpha\beta[(\alpha\beta)^{\alpha+1} k_0^{\alpha^2}]^\alpha = (\alpha\beta)^{1+\alpha+\alpha^2} k_0^{\alpha^3}$$

It must be then that the solution is

$$k_t = (\alpha\beta)^{\sum_{i=0}^{t-1} \alpha_i} k_0^{\alpha^t}$$

Taking the logarithm

$$log(k_t) = \sum_{i=0}^{t-1} \alpha_i log(\alpha\beta) + \alpha^t log(k_0)$$

$$\sum_{t=0}^{\infty} \beta^t log(k_t) = \sum_{t=0}^{\infty} \beta^t \{ \sum_{i=0}^{t-1} \alpha_i log(\alpha\beta) + \alpha^t log(k_0) \}$$

$$\sum_{t=0}^{\infty} \beta^t log(k_t) = log(\alpha\beta) \sum_{t=0}^{\infty} \beta^t \sum_{i=0}^{t-1} \alpha_i + \frac{log(k_0)}{1-\alpha\beta}$$

$$\sum_{t=0}^{\infty} \beta^t log(k_t) = log(\alpha\beta) \sum_{t=0}^{\infty} \beta^t (\frac{1-\alpha^t}{1-\alpha}) + \frac{log(k_0)}{1-\alpha\beta} = log(\alpha\beta)[\sum_{t=0}^{\infty} \frac{\beta^t}{1-\alpha} - \sum_{t=0}^{\infty} \frac{\beta^t \alpha^t}{1-\alpha}] + \frac{log(k_0)}{1-\alpha\beta}$$

$$\sum_{t=0}^{\infty} \beta^t log(k_t) = log(\alpha\beta)[\frac{1}{(1-\alpha)(1-\beta)]} - \frac{1}{(1-\alpha)(1-\alpha\beta)}] + \frac{log(k_0)}{1-\alpha\beta}$$

$$\sum_{t=0}^{\infty} \beta^t log(k_t) = log(\alpha\beta)[\frac{\beta}{(1-\alpha\beta)(1-\beta)]}] + \frac{log(k_0)}{1-\alpha\beta}$$

Example 7. Substituting what we found on V

$$V(k_0) = log(\alpha\beta)[\frac{\alpha\beta}{(1-\alpha\beta)(1-\beta)]}] + \alpha\frac{log(k_0)}{1-\alpha\beta} + \frac{log((1-\alpha\beta))}{1-\beta}$$

The former equation is the answer, but note the following: The solution can be reedited as

$$V(k_0) = A + Blog(k_0)$$

Where

$$A = \frac{log(1-\alpha\beta)}{(1-\beta)} + \frac{\alpha\beta log(\alpha\beta)}{(1-\beta)(1-\alpha\beta)}$$

$$B = \frac{\alpha}{1-\alpha\beta}$$

In fact this motivates one of the standard procedures to find the solution to the Bellman Equation, guessing some functional form alike the utility form. As in differential equations a lineal solution must be used at first attempt, nevertheless this may fail. Then a quadratic can form can be tested and so on. In the final part of the introduction to Dynamic Programming we are going to review 3 methods that are available to solve the Bellman Equation.

2.3.3 Stochastic Dynamic Programming

There can be many assumptions that can be made about the nature of the stochastic behavior of the model. In this section we only are going to examine one case. The technology shocks affect the model.

$$y_t = z_t f(k_t)$$

Where $\{z_t\}$ is a sequence of random variables i.i.d. (a stochastic Process) and $f(.)$ is defined as in the previous subsection.

Then the feasible constrain is

$$k_{t+1} + c_t \leq z_t f(k_t) \quad (1A)$$

$$c_t, k_{t+1} \geq 0 \ \ all\ t\ \ all\ \ z_t$$

The key assumption of this theory is that the economy orders stochastic sequences of consumption according to expected utility

$$E[u(c_0, c_1, ...)] = E[\sum_{t=0}^{\infty} \beta^t U(c_t)] \quad (2A)$$

Where $E[.]$ is the expected value of the distribution of probability of $\{c_t\}_{t=0}^{\infty}$

The Social planner then faces

$$Max \ E[\sum_{t=0}^{\infty} \beta^t U(c_t)]$$

$$s.t.$$

$$k_{t+1} + c_t \leq z_t f(k_t)$$

$$c_t, k_{t+1} \geq 0 \ \ all \ t \ \ all \ \ z_t$$

Before go further lets explain the order of events:

At the begin of period t the actual value of z_t is observed. Then (k_t, z_t) is the state of the economy. The pair (k_t, z_t) and the total of output $z_t f(k_t)$ is known and the planner chooses the consumption c_t and at the end of the period k_{t+1} is accumulated.

As before we can think that the planner chooses besides (c_0, k_1) an infinite number of sequences, $\{(c_t, k_{t+1})\}_{t=1}^{\infty}$, nevertheless in the stochastic case these are not sequences of numbers but contingency plans. In particular think c_t and k_{t+1} to be contingent to the realization of $z_1, z_2, ...z_t$

Technically the social planner chooses sequences of functions, where the t_{th} function in the sequence has as argument the history of $(z_1, z_2, ..z_t)$ shocks that happened in the time between the plan occurs and is realized.

Then the feasible set of the planner is the set of pairs (c_0, k_1) and the sequences of functions $\{[c_t(.), k_{t+1}(.)]\}_{t=1}^{\infty}$, that satisfy (1A) every period and every realizations of the shocks.

Now the problem in sequences is complex, however we can change the problem into a problem of dynamic programming. Let $V(k, z)$ be the maximized value of (2A), with initial values of (k, z). The election of $(c, g(k))$ consumption and end of the period capital gives the current utility $U(c)$ and impacts the system in the next period.

The next period the maximum value of utility is $V(g(k), z')$, where z' is decided by nature. Nevertheless today we do not know that value. Today we have $\beta E[V(g(k), z')]$. According to this the Bellman equation must be

$$V(k, z) = max_{0 \leq g(k) \leq f(k)} \{U[zf(k) - g(k, z)] + \beta E[V(g(k, z')]\}$$

The result of this problem leads to $g(k, z)$ as a function of the state (k, z) a very similar to the deterministic case.

In fact to characterize the optimum we use the first order condition

$$U'[zf(k) - g(k, z)] = \beta E\{V[g(k, z), z']\}$$

Then the optimum policy function has state variables k and z. Implying that the capital is given by the following stochastic difference equation $k_{t+1} = g(k_t, z_t)$ where the $\{z_t\}$ are i.i.d.

2.3.4 Methods for Dynamic Programming

There are three ways to solve a Bellman Equation:

- Guessing a functional form
- Analytical iteration
- Numerical iteration

Without a doubt the easiest way is to test a solution to the problem. Nevertheless how to guess is complicated, but to the problems relevant to the course a functional form close to the objective function will be a good start.

Example 8. Test $V(k_0) = A + Blog(k_0)$ to the problem presented in example 3

We know that the problem can be written as

$$V(k_0) = \{log[k_0^\alpha - k_1] + \beta(A + Blog(k_1)]\}$$

And we know that the First order conditions lead to

$$\frac{1}{\kappa_0^\alpha - k_1} = \beta B \frac{1}{k_1}$$

Solving for k_1

$$k_0^\alpha = \frac{k_1}{\beta B} + k_1$$

$\frac{\beta B k_0^\alpha}{1+\beta B} = k_1$

Imputing the value of k_1 on the Bellman Equation

$$V(k_0) = \{log[k_0^\alpha - \frac{\beta B k_0^\alpha}{1+\beta B}] + \beta[A + Blog(\frac{\beta B k_0^\alpha}{1+\beta B})]\}$$

Rearranging

$$V(k_0) = \{log[\frac{k_0^\alpha(1+\beta B)}{1+\beta B} - \frac{\beta B k_0^\alpha}{1+\beta B}] + \beta[A + Blog(\frac{\beta B k_0^\alpha}{1+\beta B})]\}$$

Or

$$V(k_0) = \{log[\frac{k_0^\alpha}{1+\beta B}] + \beta[A + Blog(\frac{\beta B k_0^\alpha}{1+\beta B})]\}$$

Or

$$V(k_0) = \{\alpha log(k_0) - log(1+\beta B) + \beta A + \beta Blog(\beta B) + \beta B\alpha log(k_0) - B\beta log(1+\beta B)\}$$

Or

$$V(k_0) = \{\alpha log(k_0)(1+\beta B) - (1+\beta B)log(1+\beta B) + \beta A + \beta Blog(\beta B)\}$$

Then from the solution we proposed

$$B = \alpha(1+\beta B)$$

Or

$$B = \frac{\alpha}{1-\alpha\beta}$$

Now imputing the value of B

$$\{log(k_0)(\frac{\alpha}{1-\beta\alpha}) - (\frac{1}{1-\alpha\beta})log(1-\alpha\beta) + \beta A + \frac{\beta\alpha}{1-\alpha\beta}log(\frac{\alpha\beta}{1-\alpha\beta})\} = A + Blog(k_0)$$

$A = (\frac{1}{1-\alpha\beta(1-\beta)})log(1-\alpha\beta) + \frac{\beta\alpha}{(1-\beta)(1-\alpha\beta)}log(\frac{\alpha\beta}{1-\alpha\beta})$

$$A = \frac{log(1-\alpha\beta)}{(1-\beta)} + \frac{\alpha\beta log(\alpha\beta)}{(1-\alpha)(1-\beta)}$$

Then the optimum policy is

$$g(k) = k_1 = \frac{\beta B k_0^\alpha}{1+\beta B}$$

Or

$$\frac{\beta(\frac{\alpha}{1-\alpha\beta})k_0^\alpha}{1+\beta(\frac{\alpha}{1-\alpha\beta})} = \alpha\beta k_0^\alpha$$

Then

$$g(k) = \alpha\beta k_0^\alpha$$

As we can see when you have an initial guess the problem is much simpler than the one conducted in the sequences form. Most of the texts books in economics uses this approach to solve a Bellman Equation, nevertheless there are some books that prefer to solve the problem in sequences by the Euler Equation.

The Euler equation is an 18 century method that helps to find the solution when is combined with a transversality condition. But as science has advanced this method is not used as often because per se does no guarantees a solution. With Dynamic Progamming is quite easy to find the Euler Equation. Rembemer that we argue that the envelope and first order condition characterize the optimum, from here we are going to obtain the Euler equation:

$$U'[f(k) - g(k)] = \beta V'[g(k)]$$

$$V'(k) = f'(k)U'[f(k) - g(k)]$$

Set $f(k) = f(k_t)$ and $g(k) = g(k_t) = k_{t+1}$ on the F.O.C. Set also $f(k) = f(k_{t+1})$ and $g(k) = g(k_{t+1}) = k_{t+2}$ on the envelope condition

$$U'[f(k_t) - k_{t+1}] = \beta V'[k_{t+1}]$$

$$V'(k_{t+1}) = f'(k_{k+t})U'[f(k_{k+t}) - k_{t+2}]$$

Then solve for $V'(k_{t+1})$ and mix the 2 conditions

$$U'[f(k_t) - k_{t+1}] = \beta f'(k_{k+t})U'[f(k_{k+t}) - k_{t+2}], \quad (3A)$$

The former equation is known as the Euler Equation, an this was what we used to find the solution to the problem in form of sequences. Then (5) was the Euler equation and (6) was the transversality condition.

2.4 Back to Macro: A simplified Version of The Model of RBC

Its time to solve the model of RBC, however even that we know how to solve for sequences problem (by Dynamic Programming) the model does not have an analytical solution. The main reason is that some parts of the model are linear and others are log linear. To solve the model we need to make some simplifications:

- There is no government
- The depreciation of the capital is100 percent
- There is no growth of the population $n = \bar{N} = 0$
- $N = 1$
- The technology does not have a tendency component $g = \bar{A} = 0$

It is important to observe that this is a general equilibrium model. By the theorems of welfare the solution to a market equilibrium in absence of externalities are equal to the solution of the social planer. Because of this fact we can use what we have learned about dynamic programming instead of solving for every market.

The problem of the social planer is then:

$$Max_{\{c_t(A^t),l_t(A^t),K_{t+1}(A^t)\}_{t=1}^{\infty}}$$

$$U = E\{\sum_{t=0}^{\infty} e^{-\rho_t}[lnc_t + bln(1-l_t)]\}$$

$$s.t.$$

$$C_t + K_{t+1} = K_t^{\alpha}[A_t L_t]^{1-\alpha}$$

$$L_t = l_t$$

$$C_t = c_t$$

$$lnA_t = \rho_A \widetilde{A_{t-1}} + \epsilon_{A,t}$$

But with what we learned from the past section, a problem in the form of sequences can be transformed in one of Dynamic Programming. The Bellman equation for this problem is

$$V(K_t, A_t) = Max_{c_t,l_t}\{ln(c_t) + bln(1-l_t)\} + e^{-\rho}E[V(K_{t+1}, A')]$$

$$s.t.$$

$$c_t + K_{t+1} = K_t^{\alpha}[A_t l_t]^{1-\alpha}$$

$$lnA_t = \rho_A \widetilde{A_{t-1}} + \epsilon_{A,t}$$

Substituting the restrictions we have that

$$V(K_t, A_t) = Max_{K_t,l_t}\{ln(K_t^{\alpha}[A_t l_t]^{1-\alpha} - K_{t+1}) + bln(1-l_t)\} + e^{-\rho}E[V(K_{t+1}, A')]$$

Now we know how to attack this problem, guess a solution and verify it. Lets try with this solution $V(K_t, A_t) = \beta + \beta_k lnK_t + \beta_A lnA_t$

$$V(K_t, A_t) = Max_{K_t,l_t}\{ln(K_t^{\alpha}[A_t l_t]^{1-\alpha} - K_{t+1}) + e^{-\rho}E[\beta + \beta_k lnK_{t+1} + \beta_A lnA'_{t+1}]$$

We know, from the ps1 that

$$E[lnA_{t+1}] = E[\rho_A \widetilde{A_t} + \epsilon_{A,t+1}]$$

$$E[lnA_{t+1}] = \rho_A E[\widetilde{A_t}] + E[\epsilon_{A,t+1}]$$

$$E[lnA_{t+1}] = \rho_A$$

An we also know that K_{t+1}**is not random then**

$$V(K_t, A_t) = Max_{K_t,l_t}\{ln(K_t^\alpha [A_t l_t]^{1-\alpha} - K_{t+1}) + e^{-\rho}\{\beta + \beta_k ln K_{t+1} + \beta_A \rho_A\}$$

Now finding the first order conditions over K_{t+1} *F.O.C.*

$$\frac{1}{K_t^\alpha [A_t l_t]^{1-\alpha} - K_{t+1}} = \frac{e^{-\rho}\beta_k}{K_{t+1}}$$

Solving For K_{t+1}

$$\frac{K_t^\alpha [A_t l_t]^{1-\alpha} - K_{t+1}}{K_{t+1}} = \frac{1}{e^{-\rho}\beta_k}$$

$$\frac{K_t^\alpha [A_t l_t]^{1-\alpha}}{K_{t+1}} = \frac{1 + e^{-\rho}\beta_k}{e^{-\rho}\beta_k}$$

$$K_{t+1} = \frac{e^{-\rho}\beta_k}{1 + e^{-\rho}\beta_k} K_t^\alpha [A_t l_t]^{1-\alpha}$$

Plugging the values of K_{t+1} **on the equation**

$$ln(K_t^\alpha [A_t l_t]^{1-\alpha} - \frac{e^{-\rho}\beta_k}{1 + e^{-\rho}\beta_k} K_t^\alpha [A_t l_t]^{1-\alpha}) + e^{-\rho}\{\beta + \beta_k ln\{\frac{e^{-\rho}\beta_k}{1 + e^{-\rho}\beta_k} K_t^\alpha [A_t l_t]^{1-\alpha}\} + \beta_A \rho_A\}$$

Then Finding β_k

$$ln(1 - \frac{e^{-\rho}\beta_k}{1 + e^{-\rho}\beta_k}) + \alpha ln(K_t) + ln([A_t l_t]^{1-\alpha}) + e^{-\rho}\{\beta + \beta_k ln(\frac{e^{-\rho}\beta_k}{1 + e^{-\rho}\beta_k}) + ln(K_t)\alpha\beta_k + \beta_k ln\{[A_t l_t]^{1-\alpha}\} + \beta_A \rho_A\}$$

Or

$$\beta_k = \frac{\alpha + e^{-\rho}}{1 - \alpha}$$

This implies that optimal policy for capital is

$$K_{t+1} = \alpha e^{-\rho} K_t^\alpha [A_t l_t]^{1-\alpha} \quad (I)$$

$$c_t + K_{t+1} = K_t^\alpha [A_t l_t]^{1-\alpha}$$

Then using (I)

$$c_t + \alpha\beta K_t^\alpha [A_t l_t]^{1-\alpha} = K_t^\alpha [A_t l_t]^{1-\alpha}$$

This implies that the optimal value of c **is**

$$c_t = (1 - \alpha) e^{-\rho} K_t^\alpha [A_t l_t]^{1-\alpha} \ (II)$$

Finally consider the first order condition of l **and plug the found values you must have that**

$$l_t = \frac{1 - \alpha}{1 - \alpha + b(1 - ae^{-\rho})} \quad (III)$$

Given the fact that the production comes from a Cobb Douglas the wages are

$$w = (1-\alpha)K^{\alpha}l^{(-\alpha)}A^{1-\alpha}$$

Also the interest rate is

$$r = \alpha\frac{[\mathrm{Al}]^{1-\alpha}}{K^{1-\alpha}}$$

$$w = (1-\alpha)\frac{Y}{l}$$

So far we have found is that the capital of tomorrow is a fraction of what is produced, in particular we could say that the capital of tomorrow is the production multiplied by the saving rate. Then $s = \alpha e^{-\rho}$. The same is for the consumption, the consumption is the fraction that is not saved from the production. Both equation (I) and (II) are sensible to technological shocks. In fact all the fluctuations are explained by this shock. Nevertheless we can tell from equation (III) that labor is constant. how it is possible to have a labor supply constant and be not affected by the wages?. An increase in the technology raises the current wages relative to the future wages, and thus rises the labor supply. But raising the amount saved it also lowers the expected interest rate, which acts to reduce labor supply. In this model the magnitude of the effect is equal.

The conclusion of the model is that the fluctuations of the economy are Pareto optimum responses to the shocks. In particular the output is

$$Y_t = K_t^{\alpha}(A_t l_t)^{1-\alpha}$$

OR

$$lnY_t = \alpha lnK_t + (1-\alpha)(lnA_t + lnl_t)$$

and we just found that $K_t = sY_{t-1}$ thus

$$lnY_t = \alpha lns + \alpha lnY_{t-1} + (1-\alpha)(lnA_t + lnl_t)$$

Or

$$lnY_t = \alpha lns + \alpha lnY_{t-1} + (1-\alpha)(\rho_A \tilde{A_{t-1}} + \epsilon_{t,A}) + (1-\alpha)(lnl_t)$$

$$lnY_t = \alpha lns + \alpha lnY_{t-1} + (1-\alpha)(\rho_A \tilde{A}_{t-1} + \epsilon_{t,A}) + (1-\alpha)(lnl_t)$$

We can see that $\alpha lnY_{t-1} + (1-\alpha)(\rho_A \tilde{A}_{t-1} + \epsilon_{t,A})$ is not deterministic. Now we can re write the previous equation as

$$ln\widetilde{Y_t} = +\alpha ln\widetilde{Y_{t-1}} + (1-\alpha)(\rho_A \tilde{A}_{t-1} + \epsilon_{t,A})$$

If we define that $ln\widetilde{Y_t}$is the difference of lnY_t and the value of αlns and$(1-\alpha)(lnl_t)$. Now note that

$$ln\widetilde{Y_{t-1}} = +\alpha ln\widetilde{Y_{t-2}} + (1-\alpha)(\rho_A \tilde{A}_{t-2} + \epsilon_{t,A})$$

OR

$$(\rho_A \tilde{A}_{t-2} + \epsilon_{t,A}) = \frac{ln\widetilde{Y_{t-1}} - \alpha ln\widetilde{Y_{t-2}}}{1-\alpha}$$

OR

$$\tilde{A_{t-1}} = \frac{ln\widetilde{Y_{t-1}} - \alpha ln\widetilde{Y_{t-2}}}{1-\alpha}$$

$$ln\widetilde{Y_t} = +\alpha ln\widetilde{Y_{t-1}} + (1-\alpha)(\rho_A \frac{ln\widetilde{Y_{t-1}} - \alpha ln\widetilde{Y_{t-2}}}{1-\alpha} + \epsilon_{t,A})$$

OR

$$ln\widetilde{Y_t} = (\rho_A + \alpha)ln\widetilde{Y_{t-1}} - \rho_A \alpha ln\widetilde{Y_{t-2}} + (1-\alpha)\epsilon_{t,A})$$

Then the movements of the output out of the normal path follow a second order auto regressive process; The combination of a positive lag and a negative one can cause an output to have a hump shape response to disturbances. For example if $\alpha = \frac{1}{3}$ and $\rho_A = .9$ and a one time sock of $\frac{1}{(1-\alpha)}$ to ϵ_A.

Then in the firs period the output will be 1.23 up of the regular value, then 1.22 in period 2, then 1.14, 1.03, .94, .84,.76 etc. Note that he value of ρ_A plays an important role given the fact that if $\varrho_A = 0$ then the shock will disappear after two periods.

Chapter 3

Nominal Rigidity, The Neokeynesians Strikes back.

Now we are going to review how economic fluctuations are explained from the point of view of the neokeynesians. The neokeynesians try to resurrect the old ideas from Keynes. The arguments against Keynes was that there was no microeconomics behind his ideas. In short the saltwater's needed to recreate the theory with microfoundations, but keeping the lessons from Keynes. If we think in Keynes the IS-LM model pops up to our minds. Lets look the neokeynesian version of the IS-LM

3.1 Neokeynesian IS-LM

Assumptions

- Time is discrete
- Firms produce only with labor.
- Aggregate labor is given by $Y = F(L)$, $F'(.) > 0$ and $F'(.) \leqq 0$
- There is no international trade and no government.
- Aggregate consumption and Aggregate Output must be equal
- There are a fixed number of households that live forever
- The households obtain utility from consumption and from holding real money balances
- The households hate to work

The Model

The objective function of the representative household is

$$U = \sum_{t=0}^{\infty} \beta^t [u(C_t) + \Gamma(\frac{M_t}{P_t}) - v(L_t)] \quad (2)$$

We assume that the households have diminishing marginal utility on consumption and on real money balances, and increasing marginal utility on labor. We are going to assume that the households have constant relative risk aversion utility functions for the consumption and labor

$$u(C_t) = \frac{C_t^{1-\theta}}{1-\theta} \quad (3)$$

$$\Gamma(\frac{M_t}{P_t}) = \frac{(\frac{M_t}{P_T})^{1-\gamma}}{1-\gamma} \quad (4)$$

There are two assets: Money that pays zero nominal rate and bonds that pays i. We can define the wealth of the household as

$$A_{t+1} = M_t + (A_t + W_tL_t - P_tC_t - M_t)(1 + i_t) \ (5)$$

Where A_t is the wealth in period t, W_tL_t is the labor income with W_t the nominal wage, P_tC_t is the consumption expenditure. This means that the wealth is composed by the bonds the household holds and the money. The house hold problem is then

$$Max$$

$$U = \sum_{t=0}^{\infty} \beta^t [\frac{C_t^{1-\theta}}{1-\theta} + \frac{(\frac{M_t}{P_T})^{1-\gamma}}{1-\gamma} - v(L_t)]$$

$$S.t.$$

$$A_{t+1} = M_t + (A_t + W_tL_t - P_tC_t - M_t)(1 + i_t)$$

Assume that the paths of P, W and i are given. From the moment we will assume that L is exogenous.

$$V(A_t) = max\{\frac{C_t^{1-\theta}}{1-\theta} + \frac{(\frac{M_t}{P_T})^{1-\gamma}}{1-\gamma} - v(L_t)\} + \beta V(A_{t+1})$$

$$V(A_t) = max\{\frac{C_t^{1-\theta}}{1-\theta} + \frac{(\frac{M_t}{P_T})^{1-\gamma}}{1-\gamma} - v(L_t)\} + \beta V(M_t + (A_t + W_tL_t - P_tC_t - M_t)(1 + i_t))$$

$$C_t^{-\theta} = \beta V'[M_t + (A_t + W_tL_t - P_tC_t - M_t)(1 + i_t)](1 + i_t)P$$

or

$$C_t^{-\theta}/P_t = \beta V'[A_{t+1}](1 + i_t)$$

and

$$V'(A_t) = \beta V'(M_t + (A_t + W_tL_t - P_tC_t - M_t)(1 + i_t))(1 + i_t)$$

$$\frac{C_t^{-\theta}}{P_t} = V'(A_t)$$

$$\frac{C_t^{-\theta}}{P_t} = \beta\frac{C_{t+1}^{-\theta}}{P_{t+1}}(1 + i_t)$$

or

$$C_t^{-\theta} = \beta C_{t+1}^{-\theta}(1 + r_t) \ (6)$$

Where $(1 + r_t) = \frac{P_t(1+i_t)}{P_{t+1}}$ is the interest rate. Note that 6 is the Euler equation.

Taking logs of the Euler equation.

$$lnC_t = lnC_{t+1} - \frac{1}{\theta}ln[(1 + r_t)\beta] \ (7)$$

Finally if we assume that the population is normalizable to one, then $C_t = Y_t$ and if we know that $ln(1 + r_t) \sim r_t$ then we have that. and we ignore $ln(\beta)$

$$lnY_t = lnY_{t+1} - \frac{1}{\theta}r_t \ (8)$$

Equation (10) is the neokeynesian is curve, as you can see there is a negative relationship with the interest rate.

If we take find the first order condition with respect M_t we have that

$$\frac{M_t^{-\gamma}}{P_t^{1-\gamma}} + \beta V'[M_t + (A_t + W_t L_t - P_t C_t - M_t)(1+i_t)] - \beta V'[M_t + (A_t + W_t L_t - P_t C_t - M_t)(1+i_t)](1+i) = 0$$

Or

$$\frac{M_t^{-\gamma}}{P_t^{1-\gamma}} = -\beta V'[M_t + (A_t + W_t L_t - P_t C_t - M_t)(1+i_t)] + \beta V'[M_t + (A_t + W_t L_t - P_t C_t - M_t)(1+i_t)](1+i)$$

Then substituting what we know from the envolvent we find that

$$\frac{M_t^{-\gamma}}{P_t^{1-\gamma}} = -\beta \frac{C_t}{P_t \beta(1+i)}^{-\theta} + \frac{C_t^{-\theta}}{P_t}$$

Simplifying yields to

$$\frac{M_t^{-\gamma}}{P_t^{1-\gamma}} = [\frac{C_t^{-\theta} i}{P_t(1+i)}]$$

Substituting the fact that consumption and production is equal

$$(\frac{M_t}{P_t})^{-\gamma} = [\frac{Y_t^{-\theta} i}{(1+i)}]$$

Finally

$$\frac{M_t}{P_t} = (\frac{1+i}{i})^{\frac{1}{\gamma}} Y_t^{\frac{\theta}{\gamma}} \quad (10)$$

This is the Neokeynesian LM curve. You can see that the right side is the demand from real balances and the left side the money supply. Then the demand for real balances depend positive on production and negative on nominal interest rate.

If we add the assumption that prices are fixed we can do the same analysis for short period perturbations as in the old IS-LM model. An increase in the monetary supply will move the LM curve to the right, increasing the product and lowering the interest rate. Note that with fixed prices the real interest rate is the same that nominal interest rate.

3.2 Price Rigidity, Wage Rigidity, and Departures from Perfect Competition in the Goods

We have found that monetary shocks in the noekeynesian IS-LM impact the real market. nevertheless this is only true if firms supply all the additional goods. We assumed that there are fixed prices, however firms could not supply the product and not meet the additional demand. Before we look at the causes of nominal rigidity lets examine the supply side of the economy.

3.2.1 The Keynes Model (1936)

Assumptions:

$W = \bar{W}$ The nominal wage is unresponsive to current period developments

The labor market has some non Walrasian feature that causes the equilibrium real Wage to be above the market-clearing level

Prices are completely flexible and firms behave as in perfect competition

Output is given by $Y = F(L)$ and $Y'(.) > 0$ and $Y''(.) \leq 0$

This means that firms must hire at the point where real wage is equal to the marginal product of labor

$$F'(l) = \frac{\bar{W}}{P} \quad (13)$$

An increase in demand rises the product through the impact on real wage. Imagine a positive monetary shock, it rises the prices of the goods and so the real wage falls and employment rises. Because wage is above the clearing price workers supply the additional labor. Then if there is more labor it must be true that there is more output. In this case we find that real wage is counter cyclical in response to aggregate demand shocks. Nevertheless this has been rejected several times empirically.

3.2.2 Sticky Prices, Flexible Wages and Competitive Labor Market

Assumptions:

$P = \bar{P}$

Wage is flexible

Labor Market is competitive, meaning that household solve the labor supply problem.

$$V(A_t) = max\{\frac{C_t^{1-\theta}}{1-\theta} + \frac{(\frac{M_t}{P_T})^{1-\gamma}}{1-\gamma} - v(L_t)\} + \beta V(M_t + (A_t + W_t L_t - P_t C_t - M_t)(1+i_t))$$

If we find the first order condition of labor we have that

$$-v'(L_t) + \beta V'(M_t + (A_t + W_t L_t - P_t C_t - M_t)(1+i_t))(1+i)W_t = 0$$

Substituting what we know

$$\frac{C_t^{-\theta}}{P_t} W_t = v'(L_t) \quad (14)$$

In Equilibrium $Y = F(L) = C$

$$\frac{F(L)^{-\theta}}{P_t} W_t = v'(L_t)$$

or

$$\frac{W}{p} = F(L)^{\theta} v'(L) \quad (15)$$

The right hand of (15) is and increasing function of L. Then it must be true that L is an increasing function of real wage.

$$L = L^s(\frac{W}{P}) \quad (16)$$

Because it is a competitive market firms will meet demand at the prevailing price as long as it does not exceed the level where marginal cost equals price. Here there is no unemployment and the model implies a pro cyclical real wage in the face of demand fluctuations. The effective labor demand is the labor demanded that depends on the amount of goods that firms are able to seal. The real wage is the intersection of the effective labor supply curve and the labor demand curve (which is vertical below the marginal cost(intersect)

Finally the model predicts a counter cyclical markup (ratio of price to marginal cost) a rise in demand, leads to a rise in costs because wage rises and marginal product of labor declines as output rises. Prices stay fixed so the ratio falls

3.2.3 Sticky Prices, Flexible Wages and Real Labor Market Imperfections

Assumptions

$P = \bar{P}$

Wage is flexible

Labor Market is imperfect, there is some non-Walrasian feature that causes real wage to remain above the level clearing demand and supply

Firms have a real-wage function

$$\frac{W}{P} = w(L) \ (17)$$

An example of this could be that firms pay more for efficiency reasons. Now the real wage is obtained by intersecting the effective labor demand curve and the real wage function. Then the important conclusion is that there could be unemployment

3.2.4 Sticky wages, Flexible Prices, and Imperfect Competition

Assumptions

$W = \bar{W}$

Prices are flexible

The goods market is imperfectly competitive.

This means that the price is a markup over marginal cost. The market function is $\mu(L)$

price is given by

$$P = \mu(L)\frac{W}{F'(L)}$$

where $\frac{W}{F'(L)} = cmg$

Then the real wage is

$$\frac{W}{P} = \frac{F'(L)}{\mu(L)}$$

If we say that μ is constant then the wage is counter cyclical because that $F'(L) < 0$. Since the wage is fixed the price must be higher when the output is higher. As in the first case there is unemployment

If we say that μ is sufficiently countercylcical that is, if the markup is sufficiently lower in booms than in recoveries. the real wage can be acyclical or procyclical. In these cases, employment continues to be determined by the effective labor demand

As you can see different views about the sources of incomplete nominal adjustment and the characteristics of labor and good markets have different implications for unemployment, real wage and markup. As a result Keynesian theories do not make strong predictions about the behavior of these variables.

3.3 Permanent Output-Inflation Tradeoff

In practice prices and wages are not fixed forever. Assume that the level at which prices ore wages are fixed is determined by what happened in the previous period. Consider the first model, instead of having $\bar{W}$, the nominal wage is proportional to the previous period price level. That is

$$W_t = AP_{t-1} \ \ A > 0 \ (20)$$

Then we know that

$$F'(L_t) = \frac{W}{P} = \frac{AP_{t-1}}{P_t}$$

Or

$$F'(L_t) = \frac{A}{1+\pi_t} \quad (21)$$

Where π is the inflation rate. Note that equation (21) implies a positive relationship between employment and inflation. That is there is a permanent trade off of inflation and product. And since output is associated to labor there is a permanent tradeoff between inflation and labor.

However in the late 60s Freedman and Phelps argued against that. They said that in the long run such relationship was impossible. They said that in the long run firms will change the way they set their prices and wages. When policymakers adopt monetary expansive measures, they permanently increase output and employment. or with this model of the supply they reduce the real wage.. Yet there is no reason for workers and firms to settle on different levels of employment just because of inflation. This means that the equilibrium wages and employment in absence of inflation prevails even with inflation. This means they are going back to the natural rate, that can only be shifted by real forces.

3.4 Th expectations augmented Philips curve

Now neither prices or wages are completely fixed. Instead higher output is assumed to be associated with higher wages and prices. Second we allow for supply shocks and now there is a more complex adjustment from the past to the future.

$$\pi_t = \pi * + \lambda(lnY_t - ln\bar{Y}_t) + \epsilon_t^s \;\; \lambda > 0 \quad (22)$$

Where $\bar{Y}_t$ is the level of output that would prevail if prices were completely flexible, the natural rate of output.

However the main difference with the others models is $\pi*$. $\pi*$ is known as the underlying inflation because is the inflation that would prevail if there is no difference between the output and the natural rate, and there were no supply shocks.

The common neokeynesian story about this term is $\pi* = \pi_{t-1}$, now there is a tradeoff between the output and the change in inflation but not a permanent trade off between inflation and output. Because any level of inflation can occur if the product equals the natural rate. However again this model fail to avoid Freedman's comments. If there is some production above the level of the natural rate, policy makers can expand the monetary supply forever. Then the story about firms adjusting their behavior would not fit. A more monetarist equation is

$$\pi_t = \pi_t^e + \lambda(lnY_t - ln\bar{Y}_t) + \epsilon_t^s \quad (23)$$

Where π^e is the expected of inflation. This means that no policymaker can expand the product many periods above the natural rate because it requires that firms will have a very low expectations of inflation. Nevertheless the data does not support this specification, then maybe a hybrid model could work.

$$\pi_t = \phi\pi_t^e + (1-\phi)\pi_{t-1} + \lambda(lnY_t - ln\bar{Y}_t) + \epsilon_t^s \;\; 0 < \phi < 1$$

3.5 Aggregate Demand, Aggregate Supply and the AS-AD diagram.

we know that the neokeynesian IS is $lnY_t = lnY_{t+1} - \frac{1}{\theta}$. and the LM curve is $\frac{M_t}{P_t} = (\frac{1+i}{i})^{\frac{1}{\gamma}} Y_t^{\frac{\theta}{\gamma}}$. Note that when we relax the assumption that prices are fixed the LM curve has some problems, because changes in P_t will move the LM in the IS-LM diagram (Y, r). One way to avoid this issues is to assume that the central bank conducts a monetary policy that makes the real interest rate an increasing function of the gap between the actual product and the natural rate.

$$r = r(lnY_t - ln\bar{Y}_t, \pi) \ r'(.) > 0 \ \ r''(.) > 0 \ (26)$$

This means that the central bank adjust the money supply to guarantee (26). This real interest relationship with the output is positive, and we will call this the MP curve. Now if we want to capture the relationship between inflation and output we can us the AS-AD diagram. where the Aggregate Supply is (6.22). And The Aggregate demand comes form the movements in the IS MP curves. In fact a rise in π moves the MP curve, the rise in the inflation increase r and Y falls

Example 9. An IS Shock

Now we have a three model equation. The IS curve, the MP curve and the AS curve. is useful to assume that $\pi* = \pi_{t-1}$, and that the MP curve is linear. Even with this assumptions the model is quite complicated to be solved. Romer makes this additional assumptions

The MP curve now does not depend on the inflation. The shock only affect the IS curve, and the shocks follow a first order autoregresive process. Finally $ln\bar{Y}_t = 0$, $lnY_t = y_t$. This assumptions yields the following system.

$$\pi_t = \pi_{t-1} + \lambda y_t$$

$$r_t = by_t \ \ b > 0$$

$$y_t = E_t[y_{t+1}] - \frac{1}{\theta} r_t + u_t^{IS} \ \ \theta > 0$$

$$\mu_t^{IS} = \rho_{IS t} \mu_{t-1}^{IS} + e_t^{IS} \ \ -1 < \rho_{IS} < 1$$

Where e_t^{IS} is a white noise.

If we combine the MP curve with the IS curve we have that

$$y_t = E_t[y_{t+1}] - \frac{1}{\theta} by_t + u_t^{IS}$$

solving for y_t

$$y_t = \frac{\theta}{\theta + b}\{E_t[y_{t+1}] + u_t^{IS}\}$$

or

$$y_t = \Phi E_t[y_{t+1}] + \Phi u_t^{IS} \ \ (31)$$

where $\Phi = \frac{\theta}{\theta+b}$. Note that our assumptions imply that $0 < \Phi < 1$

Note that

$$y_{t+1} = \Phi E[y_{t+2}] + \Phi \mu_{t+1}^{IS}$$

Taking expectations on bot sides

$$E[y_{t+1}] = \Phi E[E[y_{t+2}]] + \Phi E[\mu_{t+1}^{IS}]$$

By the law iterated expectations

$$E_t[y_{t+1}] = \Phi E_t[y_{t+2}] + \Phi E_t[\mu_{t+1}^{IS}]$$

$$E_t[\mu_{t+1}^{IS}] = \rho_{ISt} E_t[\mu_t^{IS}] + E_t[e_{t+1}^{IS}]$$

$$E_t[\mu_{t+1}^{IS}] = \rho_{ISt} E_t[\mu_t^{IS}]$$

As the expectation is took for period t

$$E_t[\mu_{t+1}^{IS}] = \rho_{ISt} \mu_t^{IS}$$

and

$$E[y_{t+1}] = \Phi E[y_{t+2}] + \Phi \rho_{ISt} \mu_t^{IS}$$

In fact this is true for

$$Y_{t+j} = \Phi E_t[y_{t+j+1}] + \Phi u_{t+j}^{IS} \;\; for\, j = 1, 2, 3...$$

Taking expectations on both sides

$$E_t[Y_{t+j}] = \Phi E_t[y_{t+j+1}] + \Phi E_t[u_{t+j}^{IS}]$$

$$E_t[u_{t+j}^{IS}] = \rho_{ISt+j} E_t[\mu_{t+j-1}^{IS}] + E_t[e_t^{IS}]$$

$$E_t[\mu_{t+j-1}^{IS}] = \rho_{ISt+j-1} E_t[\mu_{t+j-2}^{IS}] + E_t[e_t^{IS}]$$

and if we assume that

$$\rho_{ISi} = \rho_{ISj} \;\; for\ all\ j, i$$

$$E[u_{t+j}^{IS}] = \rho_{is}^{j} u_t^{is}$$

Then

$$E[Y_{t+j}] = \Phi E_t[y_{t+j+1}] + \Phi \rho_{is}^{j} u_t^{is}$$

With this information we can go back to equation (31)

$$y_t = \Phi[\Phi E_t[y_{t+2}] + \Phi \rho_{is} \mu_t^{is}] + \Phi u_t^{IS}$$

$$y_t = \Phi[\Phi[E_t[y_{t+3}] + \Phi \rho_{IS}^2 \mu_t^{is}] + \Phi \rho_{IS} \mu_t^{is}] + \Phi u_t^{IS}$$

$$y_t = \Phi^2 E_t[y_{t+3}] + \Phi^2 \rho_{IS}^2 \mu_t^{is} + \Phi \rho_{is} \mu_t^{is} + \Phi u_t^{IS}$$

$$= (\Phi + \Phi^2 \rho_{IS} + \Phi^3 \rho_{IS}^2 ...) \mu_t^{is} + lim_{n \to \infty} \Phi^n E_t[y_{t+n}]$$

$$= \frac{\Phi}{1 - \Phi \rho_{IS}} \mu + lim_{n \to \infty} \Phi^n E_t[y_{t+n}]$$

If we assume that the limit converges to zero then

$$y_t = \frac{\Phi}{1 - \Phi \rho_{IS}} \mu_t^{IS}$$

or

$$y_t = \frac{\theta}{\theta + b - \theta \rho_{IS}} \mu_t^{IS} \;\; (35)$$

Th former expression show how different forces influence how shocks to demand affect output. a more aggressive monetary policy (a higher b) dampens the effects on the IS curve.

Substituting (35) in the supply

$$\pi_t = \pi_{t-1} + \frac{\theta\lambda}{\theta + b - \theta\rho_{IS}}\mu_t^{IS} \quad (36)$$

Thus inflation can be stabilized, if the shocks on the IS are serially correlated then the inflation is serially correlated.

Microeconomics Foundations of Incomplete Nominal Adjustment

Its time to argument why nominal changes can have effect on real variables. The neokeynesian need to prove that the monetary policy is indeed effective. Why prices remain for some periods fixed?, this works only to prices or wages also?

3.6 A Model of Imperfect Competition and Price-Setting

Assumptions

There is a continuum differentiated goods indexed by $i \in [0,1]$ Each good is produced by a singe firm with monopoly rights to the production

Firm $i's$ production is

$$Y_i = L_i \quad (37)$$

L_i is the amount of labor it hires.

Firms hire labor in competitive market, and sell the product in imperfectly competitive market

Now firms can set their prices freely

The firms are owned by the households so the profits they earn accrue to the households

We normalize the number of households to 1.

The utility of the house hold is

$$U = C - \frac{1}{\gamma}L^{\gamma} \quad \gamma > 1 \ (38)$$

C is not the households consumption of all goods. If it were, goods wold be perfect substitutes for one another and firms cannot have market power.

C is an index of household consumption of the individual goods,

$$C = [\int_{i=0}^{1} C_i^{(\eta-1)/\eta}]^{\eta/(\eta-1)} \quad \eta > 1 \ (39)$$

In this model there is no government or investment or international trade

$$Y \equiv C$$

Households choose labor to supply and their purchases on consumptions, taking as given the wage, prices of goods, and profits from firms.

Firms chose the prices and the amounts of labor they hire to produce.

Finally to add money to the model we say that

$$Y = \frac{M}{P}$$

This is a simplified version of the AD.

3.6.1 The households behavior

Consider a household that expends S

Then

$$\mathcal{L} = [\int_{i=0}^{1} C_i^{(\eta-1)/\eta}]^{\eta/(\eta-1)}di + \lambda[S - \int_{i=0}^{1} C_i P_i di]$$

Then the first order condition with respect C_i

$$\eta/(\eta-1)[\int_{j=0}^{1} C_j^{(\eta-1)/\eta}]^{1/(n-1)}\frac{\eta-1}{\eta}C_i^{-1/\eta} = \lambda P_i$$

Note that the only therms that depend on i are C_i and P_i so it must be true that

$$C_i = AP_i^{-\eta}$$

now subtitling in the budget constrain

$$S = \int_{i=0}^{1} C_i P_i di$$

$$S = \int_{j=0}^{1} AP_j^{1-\eta}dj$$

Solving for A

$$A = \frac{S}{\int_{j=0}^{1} P_j^{1-\eta}dj}$$

Then

$$C_i = \frac{S}{\int_{j=0}^{1} P_j^{1-\eta}dj}P_i^{-\eta}$$

And sustitutin in C

$$C = [\int_{i=0}^{1} \left(\frac{S}{\int_{j=0}^{1} P_j^{1-\eta}dj}P_i^{-\eta}\right)^{\frac{\eta-1}{\eta}} di]^{\eta/(\eta-1)}$$

or

$$C = \frac{S}{\int_{j=0}^{1} P_j^{1-\eta}dj}[\int_{i=0}^{1} \left(P_i^{-\eta}\right)^{\frac{\eta-1}{\eta}} di]^{\eta/(\eta-1)}$$

$$C = \frac{S}{\int_{j=0}^{1} P_j^{1-\eta}dj}[\int_{i=0}^{1} \left(P_i^{1-\eta}\right) di]^{\eta/(\eta-1)}$$

The last equation tell us that when households behave optimally, the cost of of obtaining one unit of C is $\int_{i=0}^{1} \left(P_i^{1-\eta}\right) di]^{\eta/(\eta-1)}$. That is the price index of the utility function

$$P = \int_{i=0}^{1} \left(P_i^{1-\eta}\right) di]^{1/(\eta-1)}$$

Then

$$C = \frac{S}{\int_{j=0}^{1} P_j^{1-\eta} dj} P^{-\eta}$$

Then Multiplying for $\frac{P_i^{-\eta}}{P_{I-\eta}}$

$$C = \frac{S}{\int_{j=0}^{1} P_j^{1-\eta} dj} \frac{P_i^{-\eta}}{P_{I-\eta}} P^{-\eta}$$

$$C_i = \frac{S}{\int_{j=0}^{1} P_j^{1-\eta} dj} P_i^{-\eta}$$

$$C = C_i \frac{P^{-\eta}}{P_i^{-\eta}}$$

Then

$$C_i = C \left(\frac{P_i}{P} \right)^{-\eta}$$

Thus the elasticity for each individual good is η.

The households other choice variable is labor, Its spending is $WL + R$ where W is the wage and R is the profit. So consumption is $\frac{(WL+R)}{P}$, then the problem of maximization is

$$Max_L \frac{(WL+R)}{P} - \frac{1}{\gamma} L^{\gamma}$$

Then the first order condition is

$$\frac{W}{P} - L^{\gamma-1} = 0$$

which implies that

$$L = (\frac{W}{P})^{1/\gamma-1}$$